



A Study of Converting Incomplete to Complete Caps in $PG(3, 17)$

T. H. Najim ^(1*) , E. B. Al-Zangana ⁽²⁾ 

^(1,2) Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq

Article information

Article history:

Received: February 23, 2026

Revised: June 15, 2026

Accepted: June 21, 2026

Available online: October 01, 2026

Keywords:

Cap

Complete Cap

Incomplete Cap

Finite Field

Group Action

$PG(3,17)$

Correspondence:

Tabark Hassan Najim.

tabarkhn2024@uomustansiriyah.edu.iq

u.iq

Abstract

This paper investigated the complete caps of different degrees in the finite three-dimensional projective space on the finite field of order seventeen, $PG(3,17)$. These caps were constructed from thirty-four caps obtained by the right action of thirty-four cyclic subgroups, from the cyclic group generated by the companion matrix, on the projective space. These caps are classified into complete and incomplete caps, and then an extension has been done with respect to the degree and size. The maximum size of caps has been studied amongst all the obtained complete caps. Also, we found the lines that partition the space $PG(3,17)$, whose number was 290 lines.

DOI: [10.33899/etpy3506](https://doi.org/10.33899/etpy3506), ©Authors, 2026, College of Education for Pure Science, University of Mosul.

This is an open access article under the CC BY 4.0 license (<http://creativecommons.org/licenses/by/4.0/>).

1. Introduction

A finite projective space $PG(n, q)$ is an n -dimensional projective space over the finite field $GF(q) = F_q$, where $q = p^h$ (p prime number and h is a natural number). Its l -subspaces are just the $(l + 1)$ -subspaces of $(n + 1)$ -dimensional vector space over the finite field F_q . For details, see [1] and [2]. In this paper, we consider the three-dimensional projective space $PG(3,17)$. The number of points in $PG(3, q)$ is $(q^3 + q^2 + q + 1)$. So, the space $PG(3,17)$ contains 5220 points. Each line in $PG(3, q)$ contains $(q + 1)$ points so, each line in the space $PG(3,17)$ contains 18 points. There exist $(q^2 + 1)(q^2 + q + 1)$ lines in $PG(3, q)$. So, there are 89215 lines in $PG(3,17)$. For general details of $PG(3, q)$, see [3] [4].

A classification of an object cap in the finite projective space is still the aim of many researchers because of the relation between objects in the finite projective spaces and coding theory; see [5] [6] [7]. There are many studies as an application to this relation in coding theory and also in cryptography theory; see [8] [9].

To our knowledge, there is no specialized study on caps of degrees higher than 2 in $PG(3,17)$ that aims to classify the caps of degrees higher than two, but there are special cases that have been studied in fields of order 7, 8, 11, 13, 16, and 23.

Here we list some of the most recent papers related to the research objective. In $PG(3,8)$, Al-Rikabi introduced new sizes of caps with different degrees higher than two [10]. Radhi in [11], studied the $m_r(3,11)$ of cap of degree $r \geq 2$, and then discussed the extension of cap according to sizes and degrees in [12]. In the same line of idea of the above sources, but for $q = 13$ and $q = 23$ some objects like caps and arcs have done many extensive studies and statistical analysis, see [13] [14]. Also, caps and spans in $PG(3,7)$ have been studied in [15]. In the case of caps of degree two, which is the most studied case because the dual code of the complete cap of degree two has Hamming distance four and covering radius two, many general results have been founded for example:

1- $m_2(3, q) = q^2 + 1$;

2- $t_2(3, q) \leq 2q + t_2(3, q), q = 2^h, h \geq 2;$

3- $t_2(3, q) \leq 3q$ for $2 \leq q \leq 17;$

4- $t_2(3, q) \leq 4q$ for $2 \leq q \leq 89;$

In the case of $q = 17,$

5- $26 \leq t_2(3, 17).$

For details about the above results, see [16] and references therein.

The main problems on (k, r) -caps in $PG(n, q)$ concern the large value of the size of (complete) caps, and very small (complete) caps. This research aims to find caps of different degrees and sizes, in particular, the complete caps. Our aim can be regarded as a continuation of the work in the papers [10], [11], [12], [13], and [14], where the action of a group on the space has been used to construct caps of different sizes and degrees, and use them as a starting step to obtain bounds of $m_r(3, 17)$.

2. Definitions and Preliminaries

Definition 2.1: [1] (Cap)

A cap in $PG(n, q)$ of size k and degree r is a set of k points such that no $r + 1$ of them are collinear. In other words, any line in the space contains at most r points of the set. It will be denoted by (k, r) -cap.

Definition 2.2: [1] (Complete Cap)

A complete cap of fixed degree is a cap that cannot be extended any further. A Complete cap is therefore a maximal with respect to the size with fixed degree. The maximum value of k for an (k, r) -cap with respect to the degree r is denoted by $m_r(n, q)$.

Definition 2.3: [1] (Incomplete Cap)

An incomplete cap of degree r is a cap that is not maximal; that is, it is possible to add more points to the set while still preserving the property that no $r + 1$ points are collinear.

Let K be a cap of degree k . An i -secant of K in $PG(n, q)$ is a line L such that $|K \cap L| = i$. The number of i -secants of K denoted by τ_i . Let Q be a point not on the (k, r) -cap. The number of i -secant of K passing through Q is denoted by $\sigma_i(Q)$. The number $\sigma_r(Q)$ of r secants is called the index of Q with respect to K . the set of all points of index i will be denoted by C_i and the cardinality of C_i denoted by c_i [1].

3. Algorithm:

The main goal of this study is to start from incomplete caps, extend them into complete caps, and then increase the degree of complete caps by adding extra points, while analyzing how many points are required at each step.

The following algorithm is used to determine whether a cap in the projective space $PG(3, 17)$ is complete or incomplete, and to extend it to a higher-degree cap when necessary. The GAP program was used to perform all computations [17].

Let S be a non-empty set of $PG(3, q)$.

Step A: Determined the degree of the cap

Determine the intersection points between lines and S to identify the degree of the cap, say r (the maximum value of intersection lines with S).

Step B: Check whether the cap is complete or incomplete by verifying that c_r equal zero or not.

Step C: Extension of the size

If the cap is incomplete, add new points from C_r to make it complete by going back to Step B.

Step D: Extension of the degree

After obtaining a complete cap from Step C, say S^+ . Then we will increase its degree by doing the following:

Determine a line, say l meet S^+ in r points. Then add a point from the set $l - l \cap S^+$ and then back to Steps B and C. Still adding points from $l - l \cap S^+$ till $r = q + 1$ (the maximum value of degree), which is equal to the number of points on the line.

Note: Each cap is constructed in both cases, complete and incomplete. If we add a point to change the degree of the cap, the obtained new cap will be an incomplete cap.

4. Caps in $PG(3, 17)$

The order of the projective space $PG(3, 17)$ is $\theta(3, 17) = 5220$; that is, $|PG(3, 17)| = 5220$. In $PG(3, 17)$, there are 89,215 lines and 307 subspaces of dimension two (projective plane).

Let $\mathcal{T} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a^{15} & a^8 & a^8 & a \end{bmatrix}$ be the companion matrix over F_{17} , where a is the primitive element of F_{17} . The points of $PG(3, 17)$

will be computed using the formula $[1, 0, 0, 0] * \mathcal{T}^i \bmod 17, i = 0, \dots, 5219$.

Note: The numeral form has been taken to represent the points of the space for simplicity, for example: $[1, 0, 0, 0] * \mathcal{T}^2 \bmod 17 \equiv [0, 1, 0, 0]$ will refer to 2.

The factors of $\theta(3, 17) = 5220$ are given bellow in two rows $R1$ and $R2$ such that $R1(i) * R2(j) = \theta(3, 17)$.

R1	2610	1740	1305	1044	870	580	522	435	348	290	261	180	174	145	116	90	87
R2	2	3	4	5	6	9	10	12	15	18	20	29	30	36	45	58	60

Lemma 4.1: There are 34 cyclic subgroups of $PGL(3,17)$ and 34 orbits of $PG(3,17)$ from the group actions.

Proof: Let $G = \langle T \rangle$ be the cyclic group generated by the companion matrix T , which is of order $\theta(3,17)$. Since we have 34 divisors of $\theta(3,17) = 5220$ as given in R1 and R2, so we have 34 subgroups of G , $G_{k_i} = \langle T^{k_i} \rangle$ of order s_i where k_i and s_i are in column i of R1 and R2. The right action of each subgroup G_{k_i} on $PG(3,17)$:

$$PG(3,17) \times G_{k_i} \rightarrow PG(3,17)$$

$$(p, T^{k_i j}) \rightarrow pT^{k_i j}, \quad j = 0, \dots, s_i - 1$$

will partitioned the space $PG(3,17)$ into k_i equivalent orbits of size s_i . Let O_{k_i} be the represented orbit. So, we have 34 orbits from the action of G_{k_i} on $PG(3,17)$. The details of these orbits are as follows:

- | | |
|--|---|
| 1. $O_2 = \{2k + 1 \mid k = 0, \dots, 2609\},$ | 18. $O_{87} = \{87k + 1 \mid k = 0, \dots, 59\},$ |
| 2. $O_3 = \{3k + 1 \mid k = 0, \dots, 1739\},$ | 19. $O_{90} = \{90k + 1 \mid k = 0, \dots, 57\},$ |
| 3. $O_4 = \{4k + 1 \mid k = 0, \dots, 1304\},$ | 20. $O_{116} = \{116k + 1 \mid k = 0, \dots, 44\},$ |
| 4. $O_5 = \{5k + 1 \mid k = 0, \dots, 1043\},$ | 21. $O_{145} = \{145k + 1 \mid k = 0, \dots, 35\},$ |
| 5. $O_6 = \{6k + 1 \mid k = 0, \dots, 869\},$ | 22. $O_{174} = \{174k + 1 \mid k = 0, \dots, 29\},$ |
| 6. $O_9 = \{9k + 1 \mid k = 0, \dots, 579\},$ | 23. $O_{180} = \{180k + 1 \mid k = 0, \dots, 28\},$ |
| 7. $O_{10} = \{10k + 1 \mid k = 0, \dots, 521\},$ | 24. $O_{261} = \{261k + 1 \mid k = 0, \dots, 19\},$ |
| 8. $O_{12} = \{12k + 1 \mid k = 0, \dots, 434\},$ | 25. $O_{290} = \{290k + 1 \mid k = 0, \dots, 17\},$ |
| 9. $O_{15} = \{15k + 1 \mid k = 0, \dots, 347\},$ | 26. $O_{348} = \{348k + 1 \mid k = 0, \dots, 2609\},$ |
| 10. $O_{18} = \{18k + 1 \mid k = 0, \dots, 289\},$ | 27. $O_{435} = \{435k + 1 \mid k = 0, \dots, 11\},$ |
| 11. $O_{20} = \{20k + 1 \mid k = 0, \dots, 260\},$ | 28. $O_{522} = \{522k + 1 \mid k = 0, \dots, 9\},$ |
| 12. $O_{29} = \{29k + 1 \mid k = 0, \dots, 179\},$ | 29. $O_{580} = \{1, 581, 1161, 1741, 2321,$ |
| 13. $O_{30} = \{30k + 1 \mid k = 0, \dots, 173\},$ | $2901, 3481, 4061, 4641\},$ |
| 14. $O_{36} = \{36k + 1 \mid k = 0, \dots, 144\},$ | 30. $O_{870} = \{1, 871, 1741, 2611, 3481, 4351\},$ |
| 15. $O_{45} = \{45k + 1 \mid k = 0, \dots, 115\},$ | 31. $O_{1044} = \{1, 1045, 2089, 3133, 4177\},$ |
| 16. $O_{58} = \{58k + 1 \mid k = 0, \dots, 89\},$ | 32. $O_{1305} = \{1, 1306, 2611, 3916\},$ |
| 17. $O_{60} = \{60k + 1 \mid k = 0, \dots, 86\},$ | 33. $O_{1740} = \{1, 1741, 3481\},$ |
| | 34. $O_{2610} = \{1, 2611\}.$ ■ |

Theorem 4.2: The 34 orbits O_i divided into complete and incomplete caps with the following details:

(i) Five complete caps of size 1740 and degree 10, size 1305 and degree 9, size 870 and degree 6, size 580 and degree 4, and size 290 and degree 2.

(ii) Twenty-nine incomplete caps of degrees between 2 and 9, and six of degree 18.

Proof: The algorithm in Section 3 is used with GAP programming to get the results depending on the orbits in Lemma 1. The results appear in the second columns of Table 1 and Table 2. ■

All 29 incomplete caps have been transformed into complete caps. After that, all the completed caps of degrees less than 18 have been extended by one degree until degree 18. All the new caps with the new degree were incomplete. Finally, these incomplete caps have been transformed into complete caps. The details of these caps are in Tables 1 and 2.

Table 1 shows the details of the five complete caps amongst the 34 caps. O_i and design as follows. The first column is the initial orbit symbol, which refers to the complete cap. The second column is the initial size of the cap. The third column is the initial degree of the cap. The fourth column contains the additional points to increase the degree of the initial cap. The fifth column is the number of additional points to the initial cap to become complete, which is denoted by $N^\#$. The sixth column is the final, the maximum size of the original cap to become complete after increasing its degree, as in the seventh column.

Table 1. Details of the complete caps and its degree extensions

O_i	Initial size	Initial degree	Points added to increase the degree	$N^\#$	Final size	Final degree
O_3	1740*	10	3	127	1868*	11
			3,5	381	2123	12
			3,5,86	623	2366	13
			3,5,86,1682	984	2728	14
			3,5,86,1682, 2534	1312	3057	15
			3,5,86,1682,2534,4418	1742	3488*	16
			3,5,86,1682,2534,4418,4647	2102	3849	17
			3,5,86,1682,2534,4418,4647,4790	3742	5220	18
O_4	1305	9	6	305	1611	10
			6, 86	545	1852	11
			6, 86, 803	795	2103	12
			6, 86, 803, 1446	1086	2359	13
			6, 86, 803, 1446, 2099	1403	2713	14
			6, 86, 803, 1446, 2099, 3195	1774	3085*	15
			6, 86, 803, 1446, 2099, 3195, 3686	2158	3470	16
			6, 86, 803, 1446, 2099, 3195, 3686,3916	2609	3922	17
O_6	870*	6	6, 86, 803, 1446, 2099, 3195, 3686, 3916, 4347	3906	5220	18
			2	92	963*	7
			2,510	277	1149*	8
			2,510,737	481	1354*	9
			2,510,737,1011	684	1558	10
			2,510,737,1011,1434	924	1799	11
			2,510,737,1011,1434,1673	1187	2063	12
			2,510,737,1011,1434,1673,2559	1478	2355	13
O_9	580*	4	2,510,737,1011,1434,1673,2559,2655	1789	2667	14
			2,510,737,1011,1434,1673,2559,2655,3278	2189	3068	15
			2,510,737,1011,1434,1673,2559,2655,3278,3930	2560	3440	16
			2,510,737,1011,1434,1673,2559,2655,3278,3930,4746	2999	3880	17
			2,510,737,1011,1434,1673,2559,2655,3278,3930,4746, 5120	4338	5220	18
			8	41	622*	5
			8,9	171	753	6
			8,9,517	352	935	7
O_9	580*	4	8,9,517,744	546	1130	8
			8,9,517,744,1382	780	1365	9
			8,9,517,744,1382,1680	994	1580	10
			8,9,517,744,1382,1680,1982	1246	1833	11
			8,9,517,744,1382,1680,1982,2662	1520	2108	12
			8,9,517,744,1382,1680,1982,2662,2996	1860	2449	13
			8,9,517,744,1382,1680,1982,2662,2996,3285	2160	2750*	14
			8,9,517,744,1382,1680,1982,2662,2996,3285,3362	2471	3062	15
O_9	580*	4	8,9,517,744,1382,1680,1982,2662,2996,3285,3362,3937	2849	3441	16
			8,9,517,744,1382,1680,1982,2662,2996,3285,336,3937,4502	3330	3923*	17
			8,9,517,744,1382,1680,1982,2662,2996,3285,336,3937, 4502, 5127	4626	5220	18

O_i	Initial size	Initial degree	Points added to increase the degree	$N^\#$	Final size	Final degree
O_{18}	290	2	2	21	312*	3
			2,510	104	396	4
			2,510,737	243	536	5
			2,510,737,1011	407	701	6
			2,510,737,1011,1375	592	887	7
			2,510,737,1011,1375,1434	795	1091	8
			2,510,737,1011,1375,1434,1673	1005	1302	9
			2,510,737,1011,1375,1434,1673,1975	1251	1549	10
			2,510,737,1011,1375,1434,1673,1975,2559	1501	1800	11
			2,510,737,1011,1375,1434,1673,1975,2559,2655	1792	2092	12
			2,510,737,1011,1375,1434,1673,1975,2559,2655,3278	2156	2457	13
			2,510,737,1011,1375,1434,1673,1975,2559,2655,3278,3355	2418	2720	14
			2,510,737,1011,1375,1434,1673,1975,2559,2655,3278,3355,3930	2720	3023	15
			2,510,737,1011,1375,1434,1673,1975,2559,2655,3278,3355,3930,4495	3096	3400	16
			2,510,737,1011,1375,1434,1673,1975,2559,2655,3278,3355,3930,4495,4746	3580	3885	17
			2,510,737,1011,1375,1434,1673,1975,2559,2655,3278,3355,3930,4495,4746,5120	4914	5220	18

Table 2 shows the details of the 29 incomplete caps amongst the 34 orbits, and the design is as follows. The second column is the initial orbit symbol, which refers to the incomplete cap. The third column is the initial size of the cap. The fourth column is the initial degree of the cap. The fifth column is the number of additional points to the initial cap to become complete, which is denoted by $N^\#$. The sixth column is the final status of the original cap, which is a complete cap.

Table 2. Details of the incomplete caps and their size extensions

No.	O_i	Initial size	Initial degree	$N^\#$	Final status Complete cap
1.	O_2	2610	18	2610	(5220,18)-cap
2.	O_5	1044	18	4176	(5220,18)-cap
3.	O_{10}	522	18	4698	(5220,18)-cap
4.	O_{12}	435	5	91	(526,5)-cap
5.	O_{15}	348	6	303	(651,6)-cap
6.	O_{20}	261	9	1012	(1273,9)-cap
7.	O_{29}	180	18	5040	(5220,18)-cap
8.	O_{30}	174	174	494	(668,6)-cap
9.	O_{36}	145	145	145	(290,2)-cap
10.	O_{45}	116	116	185	(301,4)-cap
11.	O_{58}	90	18	5130	(5220,18)-cap
12.	O_{60}	87	4	224	(311,4)-cap
13.	O_{87}	60	6	586	(646,6)-cap
14.	O_{90}	58	2	232	(290,2)-cap
15.	O_{116}	45	9	1213	(1258,9)-cap
16.	O_{145}	36	18	5184	(5220,18)-cap
17.	O_{174}	30	6	603	(633,6)-cap
18.	O_{180}	29	2	35	(64,2)-cap
19.	O_{261}	20	2	43	(63,2)-cap
20.	O_{290}	18	18	5202	(5220,18)-cap

No.	O_i	Initial size	Initial degree	$N^\#$	Final status Complete cap
21.	O_{348}	15	3	151	(166,3)-cap
22.	O_{435}	12	6	620	(632,6)-cap
23.	O_{522}	10	2	53	(63,2)-cap
24.	O_{580}	9	9	1262	(1271,9)-cap
25.	O_{870}	6	6	633	(639,6)-cap
26.	O_{1044}	5	2	55	(60,2)-cap
27.	O_{1305}	4	2	58	(62,2)-cap
28.	O_{1740}	3	3	166	(169,3)-cap
29.	O_{2610}	2	2	61	(63,2)-cap

As appears in Table 2, there are 22 complete caps of degree less than 18 from each incomplete cap O_{12} , O_{15} , O_{20} , O_{30} , O_{36} , O_{45} , O_{60} , O_{87} , O_{90} , O_{116} , O_{174} , O_{180} , O_{261} , O_{348} , O_{435} , O_{522} , O_{580} , O_{870} , O_{1044} , O_{1305} , O_{1740} , O_{2610} . These caps have been extended to their degree (the new cap will be incomplete) until 18, and then their status will be changed from an incomplete cap to a complete cap. The details in Table 3 are designed as follows:

The table partitions into 22 blocks. The first row of each block contains the symbol of the initial incomplete cap, the second and fourth rows contain the degrees of the complete caps (denoted by $DE^\#$) and under each of them, the size of the complete cap (denoted by $S^\#$) in the third and fifth rows, respectively.

Table 3. Degree extension and size extension to the complete caps in Table 2

1.	O_{12}								
$DE^\#$	2	3	4	5	6	7	8	9	10
$S^\#$	—	—	—	526	679	875	1065	1288	1522
$DE^\#$	11	12	13	14	15	16	17	18	
$S^\#$	1784	2067	2397	2702	3023	3400	3911	5220	
2.	O_{15}								
$DE^\#$	2	3	4	5	6	7	8	9	10
$S^\#$	—	—	—	—	651	851	1061	1299	1535
$DE^\#$	11	12	13	14	15	16	17	18	
$S^\#$	1824	2072	2402	2700	2995	3402	3877	5520	
3.	O_{20}								
$DE^\#$	2	3	4	5	6	7	8	9	10
$S^\#$	—	—	—	—	—	—	—	1273	1524
$DE^\#$	11	12	13	14	15	16	17	18	
$S^\#$	1807	2083	2407	2719	2988	3403	3893	5220	
4.	O_{30}								
$DE^\#$	2	3	4	5	6	7	8	9	10
$S^\#$	—	—	—	—	668	856	1052	1264	1547
$DE^\#$	11	12	13	14	15	16	17	18	
$S^\#$	1796	2089	2402	2701	2951	3383	3874	5220	
5.	O_{36}								
$DE^\#$	2	3	4	5	6	7	8	9	10
$S^\#$	290	228	336	488	666	848	1053	1281	1533
$DE^\#$	11	12	13	14	15	16	17	18	
$S^\#$	1796	2112	2428	2705	2957	3399	3881	5220	
6.	O_{45}								

$DE^\#$	2	3	4	5	6	7	8	9	10
$S^\#$	—	—	301	468	646	831	1057	1261	1520
$DE^\#$	11	12	13	14	15	16	17	18	
$S^\#$	1788	2100	2410	2704	2938	3389	3889	5220	
7.	O_{60}								
$DE^\#$	2	3	4	5	6	7	8	9	10
$S^\#$	—	—	311	466	644	840	1045	1270	1523
$DE^\#$	11	12	13	14	15	16	17	18	
$S^\#$	1782	2097	2400	2711	2928	3394	3882	5220	
8.	O_{87}								
$DE^\#$	2	3	4	5	6	7	8	9	10
$S^\#$	—	—	—	—	646	831	1052	1268	1520
$DE^\#$	11	12	13	14	15	16	17	18	
$S^\#$	1777	2114	2399	2693	2930	3389	3870	5220	
9.	O_{90}								
$DE^\#$	2	3	4	5	6	7	8	9	10
$S^\#$	290	171	306	462	633	829	1046	1263	1512
$DE^\#$	11	12	13	14	15	16	17	18	
$S^\#$	1779	2119	2408	2687	2904	3384	3871	5220	
10.	O_{116}								
$DE^\#$	2	3	4	5	6	7	8	9	10
$S^\#$	—	—	—	—	—	—	—	1258	514
$DE^\#$	11	12	13	14	15	16	17	18	
$S^\#$	1769	2112	2401*	2695	2912	3375	3863	5220	
11.	O_{174}								
$DE^\#$	2	3	4	5	6	7	8	9	10
$S^\#$	—	—	—	—	633	824	1051	1251	1517
$DE^\#$	11	12	13	14	15	16	17	18	
$S^\#$	1769	2128	2390	2702	2904	3383	3873	5220	
12.	O_{180}								
$DE^\#$	2	3	4	5	6	7	8	9	10
$S^\#$	64	174	302	464	637	834	1048	1264	1515
$DE^\#$	11	12	13	14	15	16	17	18	
$S^\#$	1770	2130	2389	2706	2898	3380	3874	5220	
13.	O_{261}								
$DE^\#$	2	3	4	5	6	7	8	9	10
$S^\#$	63	171	305	465	640	836	1047	1253	1509
Degree	11	12	13	14	15	16	17	18	
$S^\#$	1775	2131	2396	2691	2892	3391	3873	5220	
14.	O_{348}								
Degree	2	3	4	5	6	7	8	9	10
$S^\#$	—	166	302	462	626	831	1060	1254	1512
Degree	11	12	13	14	15	16	17	18	

$S^\#$	1772	2134	2383	2699	2905	3378	3865	5220	
15.	O_{435}								
Degree	2	3	4	5	6	7	8	9	10
$S^\#$	—	—	—	—	632	835	1054	1261	1507
Degree	11	12	13	14	15	16	17	18	
$S^\#$	1771	2140	2386	2689	2878	3379	3866	5220	
16.	O_{522}								
Degree	2	3	4	5	6	7	8	9	10
$S^\#$	63	170	308	467	640	823	1057	1260	1515
Degree	11	12	13	14	15	16	17	18	
$S^\#$	1768	2141	2386	2691	2881	3382	3864	5220	
17.	O_{580}								
Degree	2	3	4	5	6	7	8	9	10
$S^\#$	—	—	—	—	—	—	—	1271	1512
Degree	11	12	13	14	15	16	17	18	
$S^\#$	1774	2144	2391	2696	2877	3378	3862	5220	
18.	O_{870}								
Degree	2	3	4	5	6	7	8	9	10
$S^\#$	—	—	—	—	639	837	1045	1269	1512
Degree	11	12	13	14	15	16	17	18	
$S^\#$	1767	2143	2390	2700	2875	3377	3862	5220	
19.	O_{1044}								
Degree	2	3	4	5	6	7	8	9	10
$S^\#$	60	165	300	461	639	836	1049	1268	1510
Degree	11	12	13	14	15	16	17	18	
$S^\#$	1782	2150*	2396	2701	2871	3379	3871	5220	
20.	O_{1305}								
Degree	2	3	4	5	6	7	8	9	10
$S^\#$	62	167	307	464	632	838	1047	1269	1510
Degree	11	12	13	14	15	16	17	18	
$S^\#$	1775	2146	2395	2692	2854	3382	3864	5220	
21.	O_{1740}								
Degree	2	3	4	5	6	7	8	9	10
$S^\#$	—	169	309	462	634	835	1057	1276	1510
Degree	11	12	13	14	15	16	17	18	
$S^\#$	1777	2142	2384	2698	2880	3377	3862	5220	
22.	O_{2610}								
Degree	2	3	4	5	6	7	8	9	10
$S^\#$	63	169	310	464	628	839	1052	1275	1510
Degree	11	12	13	14	15	16	17	18	
$S^\#$	1775	2145	2384	2698	2880	3377	3862	5220	

In [18], Joudah proved that in $PG(3, q)$, there is an 1-space of cycle $q^2 + 1$ less than $\theta(3, q)$, and there are q cycle of lines of length $\theta(3, q)$. So, the following theorem in $PG(3, 17)$ is deduced.

Theorem 4.3: In the space $PG(3,17)$,

(i) There are 17-cycles of lines of order 5220;

(ii) There is one cycle of lines of order 290 which form a partition of the space.

Proof: The following lines l_i are founded using the GAP programming.

$l_1 = \{1, 2, 510, 737, 1011, 1375, 1434, 1673, 1975, 2559, 2655, 2989, 3278, 3355, 3930, 4495, 4746, 5120\}$,
 $l_2 = \{1, 3, 32, 84, 341, 947, 1379, 1680, 2237, 2532, 2942, 3164, 3569, 3725, 3734, 4416, 4645, 4788\}$,
 $l_3 = \{1, 4, 359, 1034, 1302, 1460, 1643, 1802, 2033, 2080, 2489, 2624, 2971, 3120, 3502, 3838, 3968, 4118\}$,
 $l_4 = \{1, 5, 115, 193, 1777, 2588, 2618, 3122, 3255, 3292, 3709, 4320, 4515, 4756, 4768, 4865, 4892, 5044\}$,
 $l_5 = \{1, 6, 86, 553, 803, 845, 1446, 1893, 2099, 2797, 2929, 3195, 3686, 3797, 3909, 3916, 4165, 4347\}$,
 $l_6 = \{1, 7, 314, 651, 1316, 1718, 1735, 3228, 3444, 3790, 3956, 4042, 4083, 4196, 5036, 5076, 5151, 5199\}$,
 $l_7 = \{1, 9, 325, 519, 529, 908, 1054, 1448, 1537, 1882, 2178, 2637, 3010, 3078, 3353, 3406, 3867, 4948\}$,
 $l_8 = \{1, 12, 1072, 1095, 1151, 1189, 1320, 2045, 2174, 2299, 2371, 2728, 2786, 3987, 4155, 4368, 4679, 4743\}$,
 $l_9 = \{1, 14, 781, 799, 1283, 1545, 1800, 2200, 2355, 2706, 2763, 2970, 3596, 3730, 3804, 4617, 4937, 5059\}$,
 $l_{10} = \{1, 15, 604, 948, 1090, 1326, 1421, 2003, 2221, 2303, 2780, 3129, 3461, 3554, 3823, 4303, 4338, 4896\}$,
 $l_{11} = \{1, 16, 218, 748, 1006, 1717, 1743, 1891, 1915, 1964, 2135, 2496, 2586, 2657, 3754, 4209, 4563, 4869\}$,
 $l_{12} = \{1, 17, 67, 1087, 2026, 2065, 2855, 2961, 3231, 3275, 3468, 3594, 3780, 3871, 3917, 4456, 4525, 5022\}$,
 $l_{13} = \{1, 20, 85, 319, 1137, 1192, 1284, 1587, 1632, 1822, 1940, 2143, 2250, 2310, 2414, 3702, 3974, 4824\}$,
 $l_{14} = \{1, 21, 174, 620, 764, 825, 924, 1328, 1796, 1899, 2370, 2432, 2589, 4573, 4607, 4995, 5100, 5133\}$,
 $l_{15} = \{1, 22, 233, 413, 903, 1225, 1365, 2011, 2074, 2597, 2695, 2882, 3060, 3096, 3854, 3921, 4238, 5009\}$,
 $l_{16} = \{1, 26, 202, 412, 739, 963, 1163, 1408, 2097, 2357, 2477, 2616, 2744, 3023, 4678, 4794, 4983, 5037\}$,
 $l_{17} = \{1, 33, 493, 735, 1128, 1414, 1827, 1927, 2421, 2703, 3124, 3175, 3660, 3947, 3990, 4361, 4938, 5025\}$,
 $l_{18} = \{1, 291, 581, 871, 1161, 1451, 1741, 2031, 2321, 2611, 2901, 3191, 3481, 3771, 4061, 4351, 4641, 4931\}$.

Then

1. $l_j = \{l_j T^{i-1}; 1 \leq i \leq 5220\}$, $j = 1, \dots, 17$ are cycles of order 5220.

2. $l_{18} = \{l_{18} T^{i-1}; 1 \leq i \leq 290\}$, is a cycle of order 290. ■

4. Conclusion

The classification of caps with degree greater than two was and still is not easy in $PG(n > 2, q)$. So, to determine the right value of is still an open problem. Regarding $PG(3,17)$, we can conclude the lower bound of $m_{r>2}(3,17)$, which has been deduced from the results in Table 1, Table 2, and Table 3, represented by the bold number with a star symbol.

$$\begin{array}{lll}
 290 = m_2(3,17), & 963 \leq m_7(3,17), & 2150 \leq m_{12}(3,17), & 3923 \leq m_{17}(3,17). \\
 312 \leq m_3(3,17), & 1149 \leq m_8(3,17), & 2457 \leq m_{13}(3,17), & \\
 580 \leq m_4(3,17), & 1365 \leq m_9(3,17), & 2750 \leq m_{14}(3,17), & \\
 622 \leq m_5(3,17), & 1740 \leq m_{10}(3,17), & 3085 \leq m_{15}(3,17), & \\
 870 \leq m_6(3,17), & 1868 \leq m_{11}(3,17), & 3488 \leq m_{16}(3,17), &
 \end{array}$$

To improve these bounded one can attempt to find another cyclic group of $PGL(4,17)$ and use the same algorithm to obtain new complete caps.

5. Acknowledgments

The authors would like to express their appreciation to University of Mustansiriyah-College of Science- Department of Mathematics for supplying the materials for this study.

6. Declarations

6.1 Approval and consent to participate

Not applicable.

6.2 Consent for publication

Not applicable.

6.3 Availability of Data and Materials

Data will be provided upon receiving a valid request.

6.4 Conflicts of interest

The authors declare that there is no conflict of interest

6.5 Funding

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

7. References

- [1] J. W. P. Hirschfeld, Projective Geometries over Finite Fields, 2nd ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, 1998. <https://doi.org/10.1093/oso/9780198502951.001.0001>.
- [2] G. Kiss and T. Szonyi, Finite Geometries, 1st ed., Chapman and Hall CRC Press, 2020. DOI: <https://doi.org/10.1201/9781315120072>.
- [3] J. W. P. Hirschfeld, Finite Projective Space of Three Dimensions:, Oxford University Press, 1985. ISBN: 0198535368, 9780198535362.
- [4] N. A. Al-seraji, A. J. Al-Rikabi, E.B. Al-Zangana., "Represent the space $PG(3, 8)$ by subspaces and sub-geometries," in *AIP Conf. Proc.*, Karbala, Iraq, 2023. DOI: <https://doi.org/10.1063/5.0114859>.
- [5] T. L. Alderson, "n-Dimensional Optical Orthogonal Codes, Bounds and Optimal Constructions," *Springer-Verlag GmbH Germany, part of Springer Nature*, vol. 30, no. 5, pp. 373-386, 2019. <https://doi.org/10.48550/arXiv.1804.07638>.
- [6] A. A. Davydov, S. Marcugini, and F. Pambianco, "On Cosets Weight Distributions of the Doubly-Extended Reed-Solomon Codes of Codimension 4," *IEEE Trans. Inform. Theory*, vol. 67, no. 8, pp. 5088-5096, 2021. DOI: [10.1109/TIT.2021.3089129](https://doi.org/10.1109/TIT.2021.3089129).
- [7] A. A. Davydov, S. Marcugini, and F. Pambianco, "On the Weight Distribution of the Costs of MDS Codes," *Advances in Mathematics of Communications*, vol. 17, no. 5, pp. 1115-1138, 2023. DOI: <https://doi.org/10.48550/arXiv.2007.08798>
- [8] V. Starodub, R. V. Skuratovskii, and S. Podpriatov, "Triangle Conics, Cubics and Possible Applications in Cryptography, Mathematics and Statistics," vol. 9, no. 5, pp. 749-759, 2021. DOI: [10.13189/ms.2021.090515](https://doi.org/10.13189/ms.2021.090515).
- [9] R. Skuratovskii and V. Osadchyy, "Criteria of Super singularity and Groups of Montgomery and Edwards Curves in Cryptography," *WSEAS Transactions on Mathematics*, vol. 19, pp. 709-722, 2021. DOI: [10.37394/23206.2020.19.77](https://doi.org/10.37394/23206.2020.19.77).
- [10] N. A. Al-seraji, A. J. Al-Rikabi, E. B. Al-Zangana, "Caps by Groups Action on the $PG(3,8)$," *Iraqi Journal of Science*, vol. 63, no. 4, pp. 1755-1764, 2022. DOI: [10.24996/ij.s.2022.63.4.33](https://doi.org/10.24996/ij.s.2022.63.4.33).
- [11] J. Sh. Radhi and E. B. Al-Zangana, "Complete (k,r) -Caps From Orbits In $PG(3,11)$," *Iraqi Journal of Science*, vol. 64, no. 1, pp. 347-353, 2023. DOI: [10.24996/ij.s.2023.64.1.32](https://doi.org/10.24996/ij.s.2023.64.1.32).
- [12] J. Sh. Radhi and E. B. Al-Zangana, "Extension of Cap by Size and Degree in the Space $PG(3,11)$," *Ibn Al-Haitham Journal for Pure and Applied Sciences*, vol. 36, no. 2, pp. 375-382, 2023. DOI: <https://doi.org/10.30526/36.2.3025>.
- [13] S. M. Attook and E. B. Al-Zangana, "Use Algebra of Group Action to Find Special Types of Caps in $PG(3,13)$," *Iraqi Journal of Science*, vol. 64, no. 8, pp. 4492-5001, 2023. DOI: [10.24996/ij.s.2023.64.8.31](https://doi.org/10.24996/ij.s.2023.64.8.31).
- [14] E. B. Al-Zangana and N. Y. K. Yahya, "Subgroups and Orbits by Companion Matrix in Three Dimensional Projective Space," *Baghdad Science Journal*, vol. 19, no. 4, pp. 805-810, 2022. DOI: <https://doi.org/10.21123/bsj.2022.19.4.0805>.
- [15] A. A. A. Raheem, "A Geometric Construction of Complete (K,r) -Cap and (K,l) -Span in $PG(3,7)$," *M.Sc. Thesis, Mosul University, Iraq*, 2000.
- [16] A. A. Davydov, G. Faina, S. Marcugini and F. Pambianco, "On sizes of complete caps in projective spaces $PG(n,q)$ and arcs in planes $PG(2,q)$," *J. Geom.*, vol. 94, pp. 31-58, 2009. DOI: [10.1007/S00022-009-0009-3](https://doi.org/10.1007/S00022-009-0009-3).
- [17] G. Group, "GAP. Reference manual," <http://www.gap-system.org/Manuals/doc/ref/manual.pdf>, 2026. <https://www.gnu.org/licenses/old-licenses/gpl-2.0.html>.
- [18] S. A. Joudah, "On Groups Action Over The Finite Projective Spaces," *MSc. Thesis, Mustansiriyah University, Iraq*, p. CH3, 2018.

دراسة حول تحويل الاغطية الغير تامة الى التامة في $PG(3, 17)$

تبارك حسن نجم⁽¹⁾, عماد بكر الزنكنة⁽²⁾

^(1,2) قسم علوم الرياضيات, كلية العلوم, الجامعة المستنصرية, بغداد, العراق

المستخلص:

بحث هذا البحث في الأغطية الكاملة ذات الدرجات المختلفة في الفضاء الإسقاط ثلاثي الأبعاد على الحقل المنتهي من الرتبة سبع عشرة $PG(3, 17)$. تم بناء هذه الأغطية من أربعة وثلاثين غطاءً تم الحصول عليها من خلال فعل لأربعة وثلاثين زمرة دورية جزئية من جهة اليمين، من الزمرة الدورية المتولدة بواسطة المصفوفة المرافقة، على فضاء الإسقاط. تم تصنيف هذه الأغطية إلى أغطية تامة وغير تامة، ثم تم توسيع نطاقها فيما يتعلق بالدرجة والحجم. تمت دراسة أقصى حجم للأغطية من بين جميع الأغطية التامة التي تم الحصول عليها. كما وجدنا الخطوط التي قسمت الفضاء $PG(3, 17)$ ، والتي بلغ عددها 290 خطاً.