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Numerical Solution for Solving Linear Fractional Differential Equations using Chebyshev Wavelets

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Abstract

In this paper, a numerical method for solving linear fractional differential equations using Chebyshev wavelets matrices has been presented. Fractional differential equations have received great attention in the recent period due to the expansion of their uses in many applications, It is difficult to find a solution to them by the analytical method due to the presence of derivatives with fractional orders. Therefore, we resort to numerical solutions. The use of wavelets in solving these equations is a relatively new method, as it was found to give more accurate results than other methods. We created Chebyshev matrices by utilizing Chebyshev sequences, where these matrices can be created in different sizes, and the larger the matrix size, The results are more accurate. Chebyshev wavelet matrices are characterized by their speed when compared to other wavelet matrices. The algorithm converts fractional differential equations into algebraic equations by using the derivative of an operational matrix of the pulsing mass of the fractional integral with Chebyshev matrices. Then, the solution is found by applying the algorithm and comparing it with the exact solution. The results are convergent with very small errors. To prove the effectiveness and applicability of the algorithm, for validation, and show how the results are close to the exact solution, several examples have been solved.

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Introduction

Fractional calculus has a wide range of applications in various fields such as science, engineering, finance, and technology [1,2]. The first concept of fractional calculus was introduced in 1695 when Leibniz wrote a letter to L'Hopital discussing the law of general differentiation and L'Hopital posed the question of what happens when the order of the derivative is 1/2. Leibniz's response was "This was an apparent paradox, but that it would lead to useful consequences in the future" [3]. For many years, fractional differential equations were primarily studied within the realm of pure mathematics. However, in recent decades, researchers have discovered their importance in a number of fields such as engineering and physics [4]. Enormous analytical, semi-analytical, and numerical methods have been developed to solve fractional differential equations [5]. While analytical solutions can be difficult to obtain, researchers found an interest in numerical solutions. Some popular methods include the Adomian decomposition method [6], the homotopy method [7], and various wavelet methods. Wavelets are a modern field in numerical solutions and include Haar wavelets [8,9], Legendre wavelets [10,11], Chebyshev wavelets [12], and Bernstein wavelets [13]. The work aims to present a method for using Chebyshev wavelets to solve fractional differential equations. The Chebyshev matrices were created by using the Chebyshev sequence, where they can be created in different sizes and the larger the matrix size, the results are more accurate. When compared to other wavelet matrices, Chebyshev wavelets are characterized by their speed. The fractional differential equations are converted into algebraic equations. Then, the solution is found and compared to the exact solution.

Preliminaries and definitions

Fractional differential equations are equations in which the derivatives appear in the form (d^{α} / dx^{α}), where α is not necessarily an integer. Several definitions of fractional integrals and derivatives have been proposed. However, Caputo and Riemann-Liouville are the most used in fractional calculus [14].

Definition 1[15]. The Riemann-Liouville fractional integral operator $J^{\alpha}$of order α is given as:

$$J^{\alpha}v(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-z)^{\alpha-1}\, v(z)\,dz\,, \qquad \alpha > o \tag{1}$$

$$J^{0}v(t)= v(t) \tag{2}$$

The properties of the operator J^{α} are given as follows:

i) $J^{\alpha}J^{\beta}v(t)=J^{\alpha+\beta}\,v(t)$ (3)

ii) $J^{\beta}J^{\alpha}\,v(t)=J^{\alpha}J^{\beta}v(t)$ (4)

iii) $J^{\alpha}t^{r}=\frac{\Gamma(r+1)}{\Gamma(\alpha+r+1)}t^{\alpha+r}$ (5)

Definition 2. [15]. The Caputo definition of a fractional derivative is given as:

$$D^{\alpha}v(t)=J^{n-\alpha}D^{n}v(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-z)^{n-\alpha-1}\, v^{(n)}(z)\,dz \tag{6}$$

Where $-1 < \alpha \le n$, $n \in \mathbb{N}$, $t > 0$

Note that

$$D^{\alpha}J^{\alpha}v(t)= v(t) \tag{7}$$

$$J^{\alpha}D^{\alpha}\,v(t) = v(t) - \sum_{k=0}^{n-1} v^{(k)}\,(0)\frac{t^{k}}{k!} \tag{8}$$

Chebyshev Wavelet and convergence of the Chebyshev wavelet

Chebyshev Wavelet

Wavelets are relatively a mathematical new area that is implemented in various fields such as numerical analysis, Image processing, signal processing, and sound pressure. They are a set of functions generated by the expansion and transformation of a specific function which is known as the "mother wavelet" **($\psi(t)$)** [15,16]

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\,\psi\left(\frac{t-b}{a}\right)\,, a, b \in \mathbb{R}, a \neq 0 \tag{9}$$

If the parameter restricts to integer values, that is $a = a_0^{-k}$, $= nb_0a_0^{-k}$, $a_0 > 1$, $b_0 > 0$, these yields:

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}}\,\psi\left(a_0^{k}t - nb_0\right) \qquad , k, n \in \mathbb{Z}\,, \tag{10}$$

for particular values of $a_0 = 2$ and $b_0 = 1$

Chebyshev wavelets $\psi_{n,m}(t) = \psi(k, n, m, t)$, have four arguments,

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}}T_m\left(2^{k}t - 2n + 1\right), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}} \\ 0 & otherwise \end{cases}\,, \tag{11}$$

where

$$T_m(t) = \begin{cases} \frac{1}{\sqrt{\Pi}} & , m = 0, \\ \sqrt{\frac{2}{\Pi}}T'_m(t) & , m > 0, \end{cases} \tag{12}$$

And $k \in \mathbb{N}, n = 1, 2, \ldots, 2^{k-1}, m = 0, 1, \ldots, M - 1,$

$T'_m(t)$ are the well-known Chebyshev polynomials of order m and satisfy the followings:

$T'_1(t) = 1$,

$T'_2(t) = t,$

$$T'_{m'+1}(t) = 2tT'_{m'}(t) - T'_{m'-1}(t)\,, \qquad m' = 1, 2, 3, \ldots \tag{13}$$

It was noticed that Chebyshev wavelets are orthogonal concerning the weight function $w_n(t) = w(2^{k}t - 2n + 1)$

A function $v(t)$ is defined over the interval [0,1) and may be extended into Chebyshev wavelets as follows:

$$\boldsymbol{v}(\boldsymbol{t}) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \boldsymbol{C}_{n,m}\, \boldsymbol{\psi}_{n,m}(\boldsymbol{t}) \tag{14}$$

Wavelet coefficients are $\boldsymbol{C}_{n,m} = (\,\boldsymbol{v}(\boldsymbol{t}), \boldsymbol{\psi}_{n,m}(\boldsymbol{t}))$ (15)

Assume that $\boldsymbol{v}(\boldsymbol{t})$ can be approximated in terms of Chebyshev wavelets as:

$$\boldsymbol{v}(\boldsymbol{t}) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} \boldsymbol{C}_{n,m}\, \boldsymbol{\psi}_{n,m}(\boldsymbol{t}) = \boldsymbol{C}^{T}\boldsymbol{\psi}(\boldsymbol{t}) = \widetilde{\boldsymbol{v}}(\boldsymbol{t}) \tag{16}$$

where $\boldsymbol{C}$ and $\boldsymbol{\psi}(\boldsymbol{t})$ are two $\mathbf{2}^{k-1}\boldsymbol{M} \times \mathbf{1}$ matrices given by:

$$\boldsymbol{C} = [\boldsymbol{C}_{1,0}, \boldsymbol{C}_{1,1}, \dots, \boldsymbol{C}_{1,M-1}, \boldsymbol{C}_{2,0}, \boldsymbol{C}_{2,1}, \dots, \boldsymbol{C}_{2,M-1}, \dots, \boldsymbol{C}_{2^{k-1},0}, \dots, \boldsymbol{C}_{2^{k-1},M-1}]^{T} \tag{17}$$

$$\boldsymbol{\psi}(\boldsymbol{t}) = [\boldsymbol{\psi}_{1,0}, \boldsymbol{\psi}_{1,1}, \dots, \boldsymbol{\psi}_{1,M-1}, \boldsymbol{\psi}_{2,0}, \boldsymbol{\psi}_{2,1}, \dots, \boldsymbol{\psi}_{2,M-1}, \dots, \boldsymbol{\psi}_{2^{k-1},0}, \dots, \boldsymbol{\psi}_{2^{k-1},M-1}]^{T} \tag{18}$$

and $\boldsymbol{T}$ indicates transposition

Let $\{\boldsymbol{t}_i\} = \{\boldsymbol{t}_i\}_{i=1}^{2^{k-1}M}$ be a set of collocation points as follows:

$$\boldsymbol{t}_i = \frac{2i-1}{2^{k}M}\,,\ \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \dots, \mathbf{2}^{k-1}\boldsymbol{M} \tag{19}$$

The Chebyshev wavelet matrix $\boldsymbol{\phi}_{k'\times k'}$ as

$$\boldsymbol{\phi}_{k'\times k'} = [\,\boldsymbol{\psi}(\boldsymbol{t}_1), \boldsymbol{\psi}(\boldsymbol{t}_2), \boldsymbol{\psi}(\boldsymbol{t}_3), \dots, \boldsymbol{\psi}(\boldsymbol{t}_{k'})] \tag{20}$$

Where $\boldsymbol{k}' = \mathbf{2}^{k-1}\boldsymbol{M}$

Let's provide the following example to illustrate the Chebyshev wavelet matrix creation when $\boldsymbol{k} = \mathbf{2}$ and $\boldsymbol{M} = \mathbf{3}$ the Chebyshev wavelet matrix is expressed as

$$\boldsymbol{\phi}_{6\times6} = \begin{bmatrix} 1.1284 & 1.1284 & 1.1284 & 0 & 0 & 0 \\ -1.0638 & 0 & 1.0638 & 0 & 0 & 0 \\ -0.1773 & -1.5958 & -0.1773 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.1284 & 1.1284 & 1.1284 \\ 0 & 0 & 0 & -1.0638 & 0 & 1.0638 \\ 0 & 0 & 0 & -0.1773 & -1.5958 & -0.1773 \end{bmatrix}$$

Convergence of the Chebyshev Wavelet

By eq. (16), $\widetilde{\boldsymbol{v}}(\boldsymbol{t})$ converge to $\boldsymbol{v}(\boldsymbol{t})$ as $\boldsymbol{k}$ approach ∞. The error can be bound, as decided by the following theorem.

Theorem: let the function $\boldsymbol{v} : [\mathbf{0}, \mathbf{1}] \to \mathbb{R}$ can be derived $\boldsymbol{n}$ times and $\boldsymbol{v} \epsilon \boldsymbol{c}^{n}[\mathbf{0}, \mathbf{1}]$. Then $\widetilde{\boldsymbol{v}}(\boldsymbol{t})$ approximate $\boldsymbol{v}(\boldsymbol{t})$ with mean error given in the form

$$\|\ \boldsymbol{v}(\boldsymbol{t}) - \widetilde{\boldsymbol{v}}(\boldsymbol{t})\ \| \le \frac{2}{2^{n(k-1)}4^{n}n!}\mathbf{sup}\left|\boldsymbol{v}^{(n)}(\boldsymbol{t})\right|.$$

Proof. Divide the [0,1] into parts $\boldsymbol{I}_{k,m} = \left[\frac{m-1}{2^{k-1}}, \frac{m}{2^{k-1}}\right], \boldsymbol{m} = \mathbf{1}, \dots, \mathbf{2}^{k-1}$, degree of $\widetilde{\boldsymbol{v}}(\boldsymbol{t}) \le \boldsymbol{n}$, $n \to \boldsymbol{v}$, $\widetilde{\boldsymbol{v}}(\boldsymbol{t}) \to \boldsymbol{v}(\boldsymbol{t})$ when $\boldsymbol{k} \to \infty$, using the maximum error estimate , Obtains

$$\|\ \boldsymbol{v}(\boldsymbol{t}) - \widetilde{\boldsymbol{v}}(\boldsymbol{t})\ \|^2 = \int_0^1 [\,\boldsymbol{v}(\boldsymbol{t}) - \widetilde{\boldsymbol{v}}(\boldsymbol{t})]^2 \boldsymbol{dx} = \sum_m \int_{I_{k,m}} [\boldsymbol{v}(\boldsymbol{t}) - \widetilde{\boldsymbol{v}}(\boldsymbol{t})]^2 \boldsymbol{dt} \le \sum_m \int_{I_{k,m}} [\boldsymbol{v}(\boldsymbol{t}) - \widehat{\boldsymbol{v}}(\boldsymbol{t})]^2 \boldsymbol{dt} \le$$
$$\sum_m \int_{I_{k,m}} [\,\frac{2}{2^{n(k-1)}4^{n}n!}\boldsymbol{sup}_{t\epsilon I_{k,m}}\left|\boldsymbol{v}^{(n)}(\boldsymbol{t})\right|]^2 \boldsymbol{dt} \le [\,\frac{2}{2^{n(k-1)}4^{n}n!}\boldsymbol{sup}_{t\epsilon[0,1]}\left|\boldsymbol{v}^{(n)}(\boldsymbol{t})\right|]^2$$

$\widehat{\boldsymbol{v}}(\boldsymbol{t})$ is the interpolating polynomial of degree $\boldsymbol{n}$. An upper bound is given by taking the square roots. When $\boldsymbol{M}$ is constant, the greater the value of $\boldsymbol{K}$, the approximation solution is more accurate.

Operational Matrix of Fractional Integration

The integration of the vector $\boldsymbol{\psi}(\boldsymbol{t})$ defined in (18) can be shown as $\int_0^t \boldsymbol{\psi}(\boldsymbol{z})\boldsymbol{dz} \approx \boldsymbol{p}\ \boldsymbol{\psi}(\boldsymbol{t})$
The fractional integration of order $\boldsymbol{\alpha}$ of vector $\boldsymbol{\psi}(\boldsymbol{t})$ in (18) can be shown as

$$(J^{\alpha}\psi)(t) \approx p^{\alpha}\psi(t) \tag{21}$$

where $\boldsymbol{p}^{\boldsymbol{\alpha}}$ is the $\boldsymbol{k}' \times \boldsymbol{k}'$ operational matrix of fractional integration of order $\boldsymbol{\alpha}$ [17].
An k'-set of Block Pulse Functions (BPFs) is defined as

$$b_i(t) = \begin{cases} 1, & \frac{i}{k'} \leq t < \frac{(i+1)}{k'}, \\ 0, & otherwise, \end{cases} \tag{22}$$

where $\boldsymbol{i} = \boldsymbol{0}, \boldsymbol{1}, \boldsymbol{2}, \dots, (\boldsymbol{k}' - \boldsymbol{1})$.

$$b_i(t)b_1(t) = \begin{cases} 0, & i \neq 1 \\ b_i(t), & i = 1 \end{cases} \tag{23}$$

$$\int_0^1 b_i(z)b_1(z)dz = \begin{cases} 0, & i \neq 1 \\ \frac{1}{k'}, & i = 1 \end{cases} \tag{24}$$

The Chebyshev wavelet matrix can also be expanded to an k'-set of (BPFs) as

$$\psi(t) = \phi_{k' \times k'} B_{k'}(t) \tag{25}$$

where $\boldsymbol{B}_{\boldsymbol{k}'}(\boldsymbol{t}) = [\boldsymbol{b}_0(\boldsymbol{t})\ \boldsymbol{b}_1(\boldsymbol{t}) \dots \boldsymbol{b}_i(\boldsymbol{t}) \dots \boldsymbol{b}_{\boldsymbol{k}'-1}(\boldsymbol{t})]^{\boldsymbol{T}}$

In Reference [18], have given the Block Pulse operational matrix of the fractional integration $\boldsymbol{F}^{\boldsymbol{\alpha}}$ as following:

$$(J^{\alpha}B_{k'})(t) \approx F^{\alpha}B_{k'}(t) \tag{26}$$

where

$$F^{\alpha} = \frac{1}{k'^{\alpha}} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \eta_1 & \eta_2 & \eta_3 & \cdots & \eta_{k'-1} \\ 0 & 1 & \eta_1 & \eta_2 & \cdots & \eta_{k'-2} \\ 0 & 0 & 1 & \eta_1 & \cdots & \eta_{k'-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \eta_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\eta_s = (s+1)^{\alpha+1} - 2s^{\alpha+1} + (s-1)^{\alpha+1}, \qquad s = 1, 2, \dots, k'-1 \tag{27}$$

Now let us derive the Chebyshev wavelet operational matrix of fractional integration

$$(J^{\alpha}\psi_{k'})(t) \approx p^{\alpha}_{k' \times k'}\psi_{k'}(t) \tag{28}$$

where matrix $\boldsymbol{p}^{\boldsymbol{\alpha}}_{\boldsymbol{k}' \times \boldsymbol{k}'}$ is called the Chebyshev wavelet operational matrix of fractional integration.
Using equations (25),(26) we obtain

$$(J^{\alpha}\psi_{k'})(t) \approx (J^{\alpha}\phi_{k' \times k'}B_{k'})(t) = \phi_{k' \times k'}(J^{\alpha}B_{k'})(t) \approx \phi_{k' \times k'}F^{\alpha}B_{k'}(t) \tag{29}$$

From equations (28),(29), we obtain

$$p^{\alpha}_{k' \times k'}\psi_{k'}(t) = p^{\alpha}_{k' \times k'}\phi_{k' \times k'}B_{k'}(t) = \phi_{k' \times k'}F^{\alpha}B_{k'}(t) \tag{30}$$

Then $p^{\alpha}_{k'\times k'}$ is given by

$$p^{\alpha}_{k'\times k'} = \phi_{k'\times k'} F^{\alpha} \phi^{-1}_{k'\times k'} \tag{31}$$

As an example the Chebyshev wavelet operational matrix of the fractional integration for $= 2$, $M = 3$, and $\alpha = 0.5$ is

$$p^{0.5}_{6\times 6} = \begin{bmatrix} 0.5116 & 0.2228 & -0.0353 & 0.4582 & -0.1067 & 0.0304 \\ -0.0579 & 0.2243 & 0.1287 & 0.0743 & -0.0449 & 0.0192 \\ -0.2120 & -0.2046 & 0.1854 & -0.2498 & 0.0501 & -0.0115 \\ 0 & 0 & 0 & 0.5116 & 0.2228 & -0.0353 \\ 0 & 0 & 0 & -0.0579 & 0.2243 & 0.1287 \\ 0 & 0 & 0 & -0.2120 & -0.2046 & 0.1854 \end{bmatrix}$$

Applications and results

The use of the proposed algorithm is illustrated by solving examples of fractional differential equations. We find the numerical solution of these equations and then compare it to the exact solution. In order to test the validity of the presented method, the absolute error and relative error between the two solutions have been calculated. The MATLAB software is performed for the computations and reported the results in terms of the largest absolute error (ME) and the relative error (Root mean square) (RMS).

ME=**Max**$\{|exact - y|\}$, y is a numerical solution. (32)

RMS=$\frac{1}{m}\sqrt{|\sum(exact - y)|^2}$ (33)

Example 1.[19]

$$D^2y(t) - 2Dy(t) + D^{\frac{1}{2}}y(t) + y(t) = 6t - 6t^2 + \frac{16}{5\sqrt{\pi}}t^{\frac{5}{2}} + t^3$$

$$y(0) = 0, \quad y'(0) = 0, \qquad 0 < t < 1,$$

The exact solution is $y(t) = t^3$

Using eq. (16), let $y(t) = C^T\,\psi(t)$

Taking the integral of the equation, where the order of integration is equal to the order of the highest derivative.

$$J^2(D^2y(t) - 2Dy(t) + D^{\frac{1}{2}}y(t) + y(t) = 6t - 6t^2 + \frac{16}{5\sqrt{\pi}}t^{\frac{5}{2}} + t^3)$$

$$C^T\left(\mathrm{I} - 2p + p^{\frac{3}{2}} + p^2\right)\psi(t) = \frac{6\Gamma(2)}{\Gamma(4)}t^3 - \frac{6\Gamma(3)}{\Gamma(5)}t^4 + \left(\frac{16}{5\sqrt{\pi}}\right)\left(\frac{\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{11}{2}\right)}\right)t^{\frac{9}{2}} + \frac{\Gamma(4)}{\Gamma(6)}t^5$$

The next step is to find both C^T ,and $y(t)$

Table 1: The results of Example 1.

Value of *K* and *M*	**ME**	**RMS**
K=2 , *M*=3	0.0107	0.0043
K=2 , *M*=4	0.0063	0.0024
K=4 , *M*=2	0.0017	6.0458e-04
K=4 , *M*=4	4.3330e-04	1.5141e-04
K=5 , *M*=5	7.0592e-05	2.4239e-05
K=5 , *M*=10	1.7753e-05	6.0605e-06
K=6 , *M*=11	3.6798e-06	1.2522e-06
K=7 , *M*=12	7.7414e-07	2.6306e-07
K=8 , *M*=10	2.7883e-07	9.4700e-08

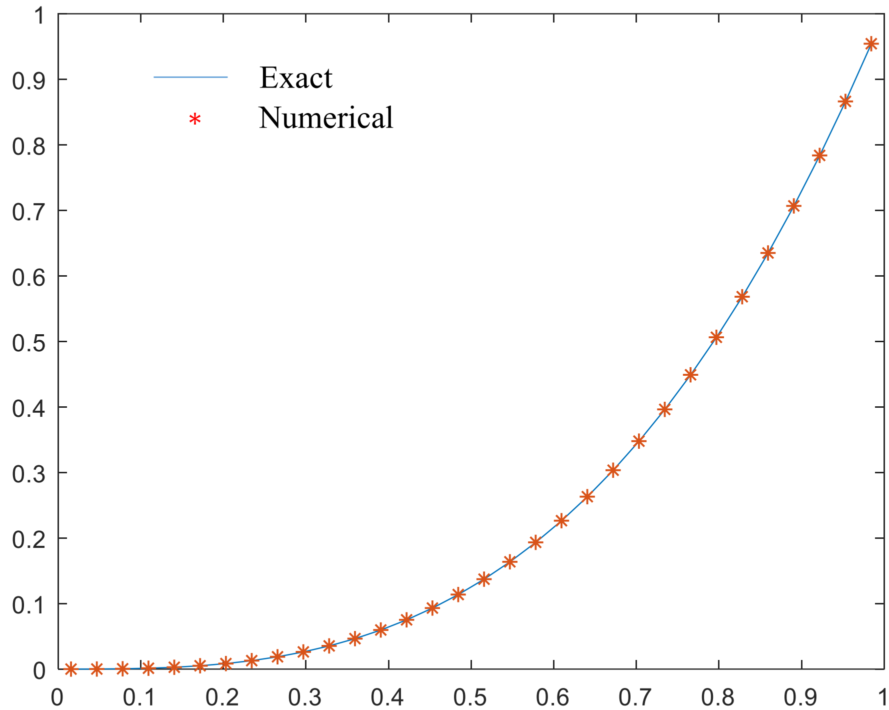

Fig 1. The exact solution and numerical solution when $k = 4, M = 4$

Example 2.[20]

$$D^{\alpha}y(t) + y(t) = \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)}t^{1-\alpha} + t^2 - t$$

$$y(0) = 0,\ \ y'(0) = 0, \quad 0 < t < 1, \qquad \alpha > 0\,,$$

The exact solution is $y(t) = t^2 - t$

Let $y(t) = C^T\ \psi(t)$

Taking the integral of the equation, where the order of integration is equal to the order of the highest derivative.

$$J^{\alpha}(D^{\alpha}y(t) + y(t) = \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)}t^{1-\alpha} + t^2 - t)$$

Using both eq. (28), and (31) yields

$$C^T(\mathrm{I} + p^{\alpha})\psi(t) = \frac{2}{\Gamma(3)}t^2 - \frac{1}{\Gamma(2)}t + \frac{\Gamma(3)}{\Gamma(3+\alpha)}t^{2+\alpha} - \frac{\Gamma(2)}{\Gamma(2+\alpha)}t^{1+\alpha}$$

Next step is to find both C^T ,and $y(t)$

Table 2: The results of Example 2, when $\alpha = 0.5$

Value of *K* and *M*	ME	RMS
K=2 , *M*=3	0.0050	0.0018
K=2 , *M*=4	0.0035	0.0010
K=4 , *M*=2	0.0014	2.6310e-04
K=4 , *M*=4	5.2470e-04	6.7969e-05
K=5 , *M*=5	1.4070e-04	1.1183e-05
K=5 , *M*=10	5.1140e-05	2.8359e-06
K=6 , *M*=11	1.6004e-05	5.9258e-07
K=7 , *M*=12	5.0344e-06	1.2543e-07
K=8 , *M*=10	2.3547e-06	4.5315e-08

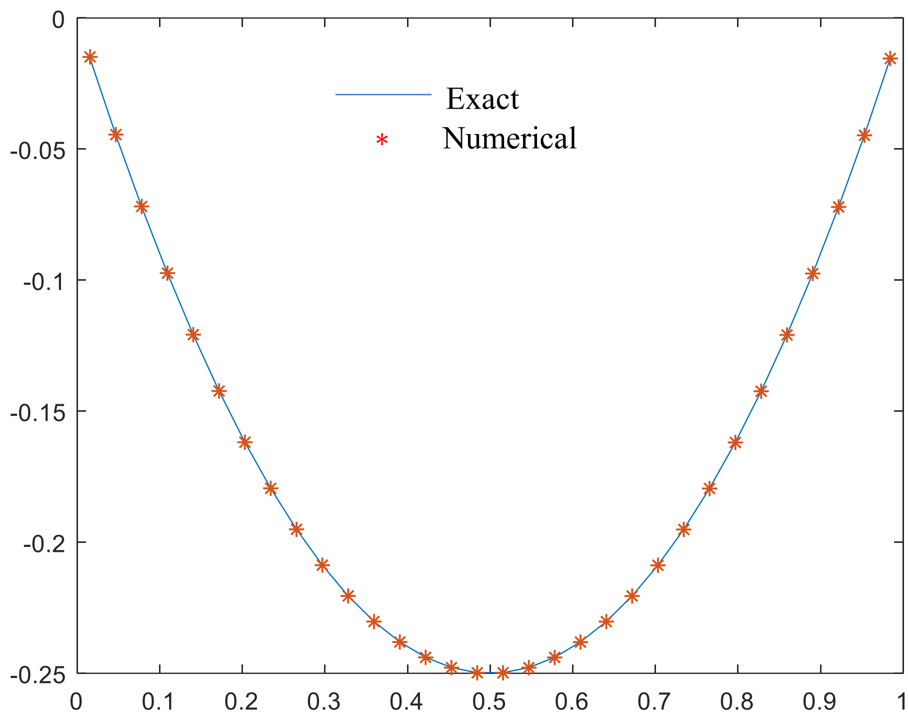

Fig 2. The exact solution and numerical solution when $k = 4, M = 4$

Example 3.[21]

$$D^2y(t) + 0.5D^{\frac{1}{2}}y(t) + y(t) = 2 + \frac{1}{\Gamma(2.5)}t^{\frac{3}{2}} + t^2,$$

$$y(0) = 0, \quad y'(0) = 0, \quad 0 < t < 1$$

The exact solution is $y(t) = t^2$

Let $y(t) = C^T\,\psi(t)$

Taking the integral of the equation, where the order of integration is equal to the order of the highest derivative.

$$J^2(D^2y(t) + 0.5D^{\frac{1}{2}}y(t) + y(t) = 2 + \frac{1}{\Gamma(2.5)}t^{\frac{3}{2}} + t^2)$$

$$C^T\left(\mathrm{I}+0.5p^{\frac{3}{2}}+p^2\right)\psi(t)=\frac{2}{\Gamma(3)}t^2+\frac{1}{\Gamma(4.5)}t^{3.5}+\frac{\Gamma(3)}{\Gamma(5)}t^4$$

Next step is to find both C^T ,and $y(t)$

Table 3: The results of Example 3.

Value of *K* and *M*	ME	RMS
K=2 , *M*=3	0.0029	0.0013
K=2 , *M*=4	0.0017	7.3514e-04
K=4 , *M*=2	4.4824e-04	1.8350e-04
K=4 , *M*=4	1.1473e-04	4.5851e-05
K=5 , *M*=5	1.8613e-05	7.3347e-06
K=5 , *M*=10	4.6745e-06	1.8336e-06
K=6 , *M*=11	9.6820e-07	3.7884e-07
K=7 , *M*=12	2.0362e-07	7.9582e-08
K=8 , *M*=10	7.3330e-08	2.8650e-08

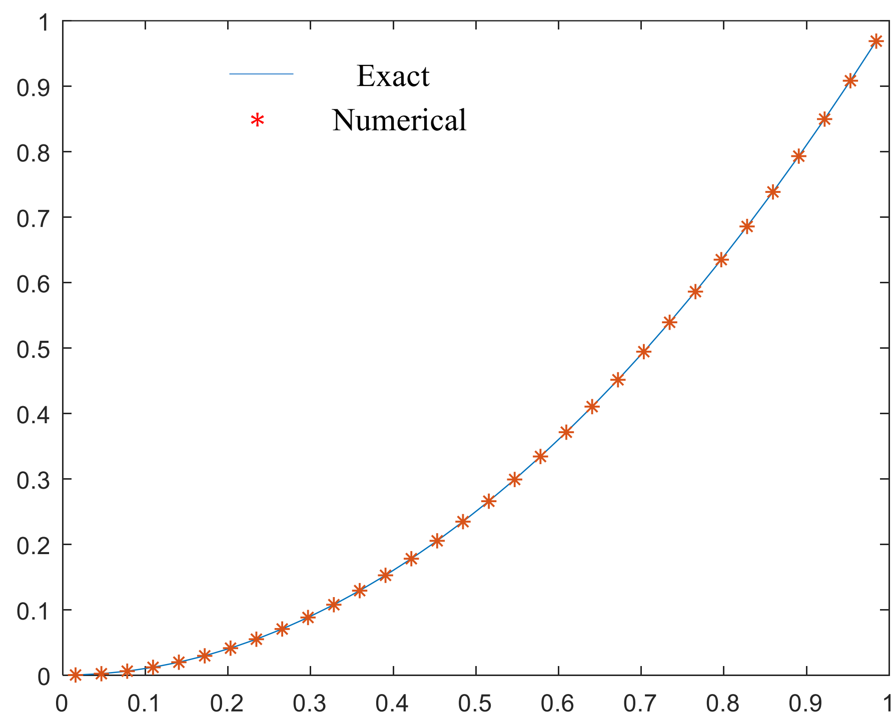

Fig 3. The exact solution and numerical solution when $k=4, M=4$

Example 4. [21]

$$D^2y(t)+\theta D^{0.3}y(t)+\beta y(t)=-12t^2+t^3\left(20+\theta\left(\frac{120}{\Gamma(5.7)}t^{1.7}-\frac{24}{\Gamma(4.7)}t^{0.7}\right)+\beta(t^2-t)\right)$$

$\beta=1$, $\theta=0.5$ $y(0)=0,\quad y'(0)=0$

The exact solution is $y(t)=t^4(t-1)$

Let $y(t)=C^T\,\psi(t)$

Taking the integral of the equation, where the order of integration is equal to the order of the highest derivative.

$$C^T(\mathrm{I} + 0.5p^{1.7} + p^2)\psi(t) = t^5 - t^4 + \frac{60}{\Gamma(7.7)}t^{6.7} - \frac{12}{\Gamma(6.7)}t^{5.7} + \frac{1}{42}t^7 - \frac{1}{30}t^6$$

The next step is to find both C^T ,and $y(t)$

Table 4: The results of Example 4

Value of K and M	ME	RMS
K=2 , M=3	2.7429e-04	1.0958e-04
K=2 , M=4	1.4815e-04	6.1799e-05
K=4 , M=2	3.9598e-05	1.6414e-05
K=4 , M=4	1.0047e-05	4.1287e-06
K=5 , M=5	1.6116e-06	6.6314e-07
K=5 , M=10	4.0302e-07	1.6597e-07
K=6 , M=11	8.3276e-08	3.4291e-08
K=7 , M=12	1.7494e-08	7.2036e-09
K=8 , M=10	6.2980e-09	2.5933e-09

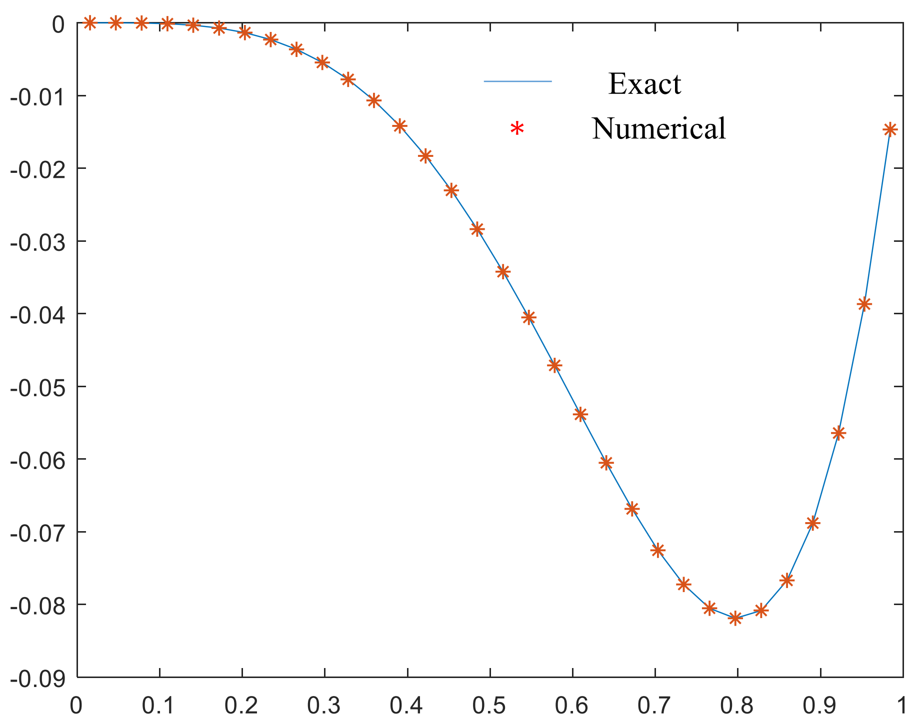

Fig 4. The exact solution and numerical solution when $k = 4, M = 4$

Example 5.[22]

$$D^2y(t) + D^{\frac{3}{2}}y(t) + y(t) = t^3 + 6t + \frac{8}{\Gamma(\frac{1}{2})}t^{\frac{3}{2}}$$

$$y(0) = 0, \quad y'(0) = 0$$

The exact solution is $y(t) = t^3$

Let $y(t) = C^T\,\psi(t)$

Taking the integral of the equation, where the order of integration is equal to the order of the highest derivative. Using both eq. (28), and (31) yields

$$C^T\left(\mathrm{I}+p^{\frac{1}{2}}+p^2\right)\psi(t)=\frac{\Gamma(4)}{\Gamma(6)}t^5+6\frac{\Gamma(2)}{\Gamma(4)}t^3+\frac{8}{\Gamma(\frac{1}{2})}\left(\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{9}{2})}\right)t^{\frac{7}{2}}$$

The next step is to find both C^T ,and $y(t)$

Table 5: The results of Example 5

Value of *K* and *M*	**ME**	**RMS**
K=2 , *M*=3	0.0056	0.0025
K=2 , *M*=4	0.0033	0.0014
K=4 , *M*=2	9.0024e-04	3.7724e-04
K=4 , *M*=4	2.3469e-04	9.6941e-05
K=5 , *M*=5	3.8662e-05	1.5864e-05
K=5 , *M*=10	9.7827e-06	4.0093e-06
K=6 , *M*=11	2.0380e-06	8.3534e-07
K=7 , *M*=12	4.3025e-07	1.7645e-07
K=8 , *M*=10	1.5523e-07	6.3689e-08

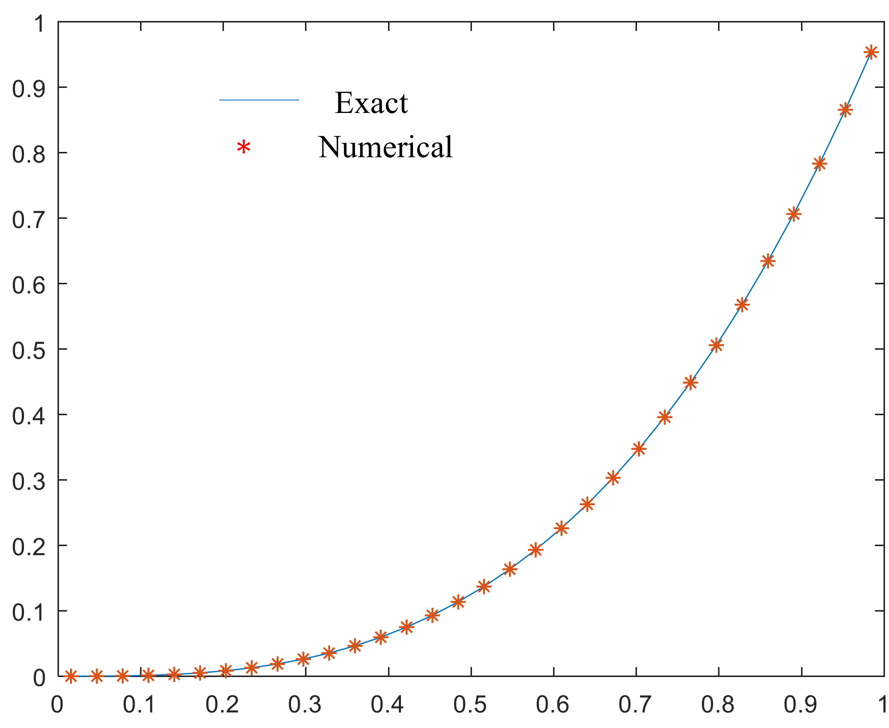

Fig 5. The exact solution and numerical solution when $k=4, M=4$

Example 6.[23]

$$D^{\frac{1}{2}}y(t)=t^2-y(t)+\frac{2}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}}$$

$y(0)=0$, $y'(0)=0$, $0<t<1$, the exact solution is $y(t)=t^2$

$$J^{\frac{1}{2}}\left(D^{\frac{1}{2}}y(t)-y(t)=t^{2}+\frac{2}{\Gamma\left(\frac{5}{2}\right)}t^{\frac{3}{2}}\right)$$
$$C^{T}\left(\mathrm{I}+p^{\frac{1}{2}}\right)\psi(t)=\frac{\Gamma(3)}{\Gamma(3.5)}t^{2.5}+\frac{2}{\Gamma(3)}t^{2}$$

The next step is to find both C^T ,and $y(t)$

Table 6: The results of Example 6

Value of *K* and *M*	ME	RMS
K=2 , *M*=3	0.0024	0.0018
K=2 , *M*=4	0.0014	0.0010
K=4 , *M*=2	3.5374e-04	2.6802e-04
K=4 , *M*=4	8.9978e-05	6.8571e-05
K=5 , *M*=5	1.4598e-05	1.1190e-05
K=5 , *M*=10	3.6732e-06	2.8250e-06
K=6 , *M*=11	7.6266e-07	5.8811e-07
K=7 , *M*=12	1.6072e-07	1.2417e-07
K=8 , *M*=10	5.7944e-08	4.4806e-08

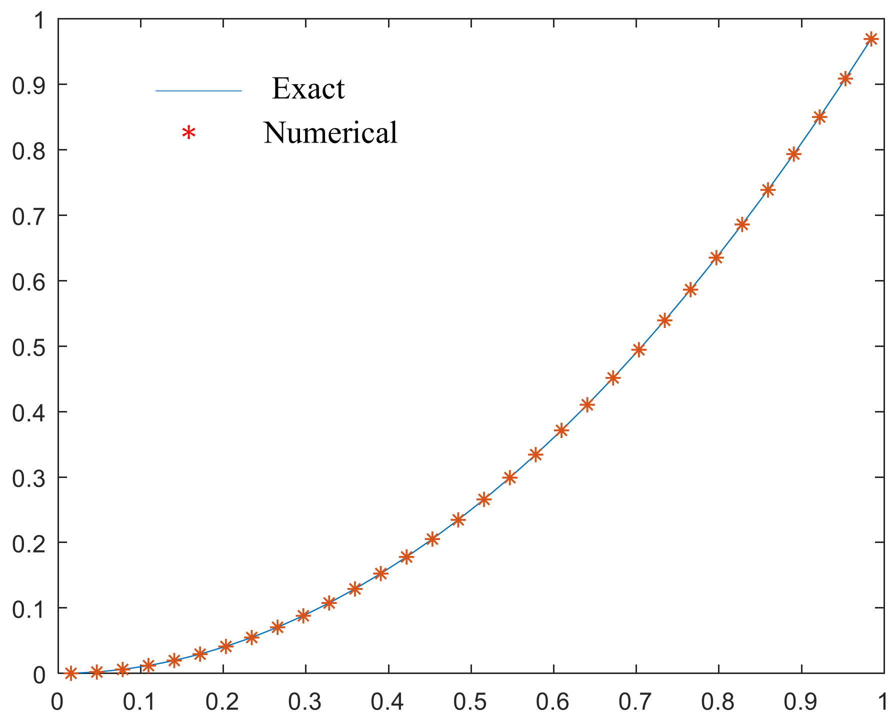

Fig 6. The exact solution and numerical solution when $k=4, M=4$

Conclusion

The Chebyshev wavelets matrix is constructed and used the block pulse operational matrix derivative of fractional integration to solve linear fractional differential equations. This was performed by converting them into algebraic equations, these equations to be solved using MATLAB software. The solution is convergent between the solution arising from the use of the algorithm and the exact solution, with a small and decreasing error rate as the matrix size increases. Six examples were presented to demonstrate the effectiveness of the proposed method.

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Conflict of interest

The author has no conflict of interest.

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الحل العددي لحل المعادلات التفاضلية الكسرية الخطية باستخدام مويجات تشيبيشيف

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الخلاصة

في هذا البحث، قدمنا طريقة عددية لحل المعادلات التفاضلية الكسرية الخطية باستخدام مصفوفات مويجات تشيبيشيف. إن المعادلات التفاضلية الكسرية لاقت اهتماماً كبيراً في الفترة الأخيرة لتوسع استخداماتها في العديد من التطبيقات ويصعب إيجاد حلها بالطريقة التحليلية لوجود مشتقات ذات رتب كسرية، لهذا نلجأ إلى الحلول العددية. يعتبر استخدام المويجات في حل هذه المعادلات طريقة حديثة نسبيًا، حيث وجد أنها تعطي نتائج أكثر دقة من الطرق الأخرى. أنشأنا مصفوفات تشيبيشيف، عن طريق استخدام متتابعات تشيبيشيف, حيث يمكن إنشاء هذه المصفوفات بأحجام مختلفة، وكلما زاد حجم المصفوفة، زادت دقة النتائج, وتتميز مصفوفات مويجات تشيبيشيف بسرعتها اذا ما قورنت بمصفوفات مويجات أخرى. تقوم الخوارزمية المقترحة بتحويل المعادلات التفاضلية الكسرية إلى معادلات جبرية باستخدام مشتق مصفوفة تشغيلية للكتلة النبضية للتكامل الكسري مع مصفوفات تشيبيشيف، ثم وجدنا الحل باستخدام الخوارزمية المذكورة وقارناه مع الحل الدقيق، النتائج متقاربة مع معدل خطأ صغير. لإثبات فعالية الخوارزمية المستخدمة وقابليتها للتطبيق وإظهار تقارب نتائجها مع الحل الدقيق، قمنا بحل ستة أمثلة