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Matrix Representation for Decomposable Solvable Six-Dimensional Lie Algebra

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Abstract

This paper extends the classification of three-dimensional connected topological proper loops L, for which the multiplication group $Mult(L)$ is a six-dimensional decomposable solvable Lie group. Building upon the significant results presented in [1], which established class two central nilpotency for these loops, we derive explicit matrix representations for the associated Lie algebras. These representations are critical for completing the classification, as they facilitate verification of structural compatibility with the conditions dictated by the Lie bracket. We identify 26 distinct families of matrix Lie groups, including 18 groups characterized by one-dimensional centers and 8 groups with two-dimensional centers. Additionally, we clarify the correspondence between these groups and their derived Lie algebras. Our findings address an existing gap in the literature by providing a systematic framework for examining the interrelationships between topological loops and their corresponding Lie groups.

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1. Introduction

The concept of an appropriate multiplication group indicated by $Mult(L)$ and an inner mapping group indicated by $Inn(L)$ associated with a loop L was first systematically presented and studied by A. Albert and R. H. Bruck. Ever since their seminal works [2], [3], extensive literature has emerged exploring the structural relationships between loops L and the associated groups $Mult(L)$ and $Inn(L)$. Notably, numerous studies have established connections between the nilpotency and solvability of loops and their corresponding multiplication and inner groups, as described extensively in references [4], [5], [6], [7], [8], [9]. M. Niemenmaa and T. Kepka [10]. Constituted criteria for G where G is a group to function as the multiplication group of a loop L. Their conditions emphasize the crucial role played by two special transversal subsets A and B, relative to a subgroup K. These transversal subsets form part of the left and right translation sets of the loop L, with the subgroup K identified as $Inn(L)$ of L. The use of permutation groups $Mult(L)$ and $Inn(L)$, as well as their related transversal subsets A and B, has been extensively addressed in literature focusing on finite loops (cf. [11], [12], [13], [14], [10], [15]). In their work [16], Nagy with Strambach thoroughly explored the topological and differentiable properties of loop L characterized by continuous and differentiable sections within Lie groups. Their findings specifically detailed topological loops and loops possessing differentiable structures on low-dimensional manifolds, as explicitly described in Part II of their research. Building upon this foundational study, the current research aims to contribute to the understanding of connected three-dimensional topological loop L, whose multiplication groups are solvable Lie groups. Previous literature [17] confirms that every connected, two-dimensional topological proper loop associated with a Lie group as its $Mult(L)$ is necessarily centrally nilpotent of class two. This result extends to connected, topological loops has dimension 3 whose $Mult(L)$ are Lie groups has a solvable property of dimension at most five, or six-dimensional solvable Lie groups (cf. [18], [19], [20], [21], [22], [23]). Furthermore, among solvable but non-nilpotent Lie groups with dimensions up to five, only decomposable groups may serve as the multiplication groups $Mult(L)$ of loop L (cf. [19]). Recently, Al-Abayechi, Ameer, and Figula, Agota [1], classified all decomposable and solvable Lie groups of dimension up to six that serve as multiplication groups $Mult(L)$ for loop L. Their research established that every connected, three-dimensional topological proper loop with a solvable Lie group of dimension at most six as its multiplication group exhibits central nilpotency of class two. Within this classification, 18 families of Lie groups with one-dimensional centers and 8 families with two-dimensional centers were identified. This paper specifically examines these associated Lie algebras, aiming to determine appropriate linear representations for the Lie groups which are simply connected, thus completing the classification process effectively.

2. Preliminaries

A non-empty set L equipped with a special operation $(x, y) \mapsto x \cdot y$ is known as a loop in case there is an identity element, where for each x in L defined the left, respectively right λ_x, respectively ρ_x, where $\lambda_x(y) = x \cdot y$ also $\rho_x(y) = y \cdot x$ are injective and surjective translations of L. Moreover, a loop L will be proper if it does not satisfy the group condition.
The permutation group is the set generated by λ_x and ρ_x for all $x \in L$ denoted by $Mult(L)$ and known as the multiplication group of L. Furthermore, the inner mapping group known as the set of L stabilizer of a special element is the identity element and denoted by $Inn(L)$.

Let $\mathfrak{S}$ be a non-empty set satisfying the group conditions, $\mathcal{K}$ is a subgroup of the group $\mathfrak{S}$, and let $\mathcal{A}$ and $\mathcal{B}$ be left two transversals to the set $\mathcal{K}$ in $\mathfrak{S}$. The sets $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{K}$-connected if they satisfy the condition that $a^{-1}b^{-1}ab \in K$ for every $a \in \mathcal{A} \wedge b \in \mathcal{B}$.

The largest normal subgroup of G contained in $\mathcal{K}$ is the core $Co_G(\mathcal{K})$ of $\mathcal{K}$ in G. If L is a loop, then the two sets $\Lambda(L)$ and $R(L)$ are $Inn(L)$-connected transversals in the group $Mult(L)$ where $\Lambda(L)$, respectively $R(L)$ is the set of all left, respectively right transversals.

The following necessary and sufficient criteria are stated in the following assertion for a group G to be the multiplication group of a loop L:

2.1 Proposition [10]:- A group $G \cong Mult(L)$ if and only if $G = \langle A, B \rangle$ and there is a subgroup $\mathcal{K}$ with $Co_G(\mathcal{K}) = 1$ and $\mathcal{K}$-connected left two transversals A and B.

In loop theory, a normal subloop is a subloop N of a loop L that satisfies the following two conditions for all $x, y \in L$, $xN = Nx$ and $(xN)(yN) = (xy)N$.

The kernel of a homomorphism α of a loop L into a loop L' is a normal subloop N of L. The center $\mathcal{Z}(L)$ of a loop L is made up of every component z which satisfies the commutative and associative identity.

If we put $\mathcal{Z}_0 = e$ with $\mathcal{Z}_1 = \mathcal{Z}(L)$ and $\mathcal{Z}_i/\mathcal{Z}_{i-1} = \mathcal{Z}(L/\mathcal{Z}_{i-1})$, then a series of L normal subloops is obtained. When $\mathcal{Z}_n = L$ and $\mathcal{Z}_{n-1}$ is a proper subloop of L, L is centrally nilpotent of class n.

If L is a topological space and the binary operations $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x \setminus y, (x, y) \mapsto y/x : L \times L \to L$ are continuous, then a loop L is said to be topological.

3. Matrix Representation

We discovered the essential conditions for those Lie algebras having dimension six, with solvable and decomposable, that can arise as the Lie algebra g of the $\boldsymbol{mult(L)}$ of L in [1], sections 3, 4, and 5. The inner mapping group of L's Lie sub algebras k were also obtained. In this section, we present a matrix Lie group representation for those Lie algebras having dimension six with solvable and decomposable. In (2005) Strugar, I., Ghanam, R., and Thompson, G., using a different strategy, created representations for all Lie algebras with dimensions of five or less in this publication (cf. [24]). We have combined these matrices to be appropriate with these decomposable Lie algebras, i.e., these Lie algebras are the sum of two proper ideals. We create in [1], §3, §4, §5, by using [25], [26], and the matrices that satisfy all Lie brackets.

3.1 The matrix representation with a discrete centre

In this case, the Lie algebras are not the $\boldsymbol{mult(L)}$ of a topological loop L of dimension 3 with connected property (cf. [1], §3). Since some cases satisfy the condition in [1] Lemma 3 c), we have to exclude these cases, i.e. we have to give a suitable matrix Lie group representation to complete the computations. The matrix representations of the Lie groups G_2, G_3 and G_4, with simply connected property, using [24] §4.

$$\mathcal{H}_2 = \begin{bmatrix} e^{x_3} & x_3 e^{x_3} & x_1 & 0 & 0 & 0 \\ 0 & e^{x_3} & x_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{x_6} & x_6 e^{x_6} & x_4 \\ 0 & 0 & 0 & 0 & e^{x_6} & x_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \mathcal{H}_3 = \begin{bmatrix} e^{x_3} & 0 & x_1 & 0 & 0 & 0 \\ 0 & e^{x_3} & x_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{x_6} & 0 & x_4 \\ 0 & 0 & 0 & 0 & e^{x_6} & x_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{H}_4 = \begin{bmatrix} e^{x_3} & 0 & x_1 & 0 & 0 & 0 \\ 0 & e^{hx_3} & x_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{x_6} & 0 & x_4 \\ 0 & 0 & 0 & 0 & e^{\frac{1}{h}x_6} & x_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, where -1 \leq h < 1,$$

we exclude these cases since the Lie subgroup set of G having dimension three and abelian, where A and B are K-connected and continuous left transversals to K in G, such that $A \cup B$ generates G, and which does not contain any non-trivial normal subgroup set of G (cf. [1], §3).

3.2 The matrix representation with a 1-dimensional centre

The 18 families of Lie groups, which are multiplication groups of topological loops L, which is 3-dimensional connected simply connected present in this instance (cf. [1], §4). Using [24] §4, we derive a matrix Lie group in a sufficiently simple form so that its Lie algebra is isomorphic to the Lie algebra g_i, $i = 1, \ldots, 18$, through differentiation and evaluation of the identity, when the matrix T is established based on six variables.

$$T_1 = \begin{bmatrix} be^{x_3} & 0 & 0 & 0 & x_4 & 0 \\ 0 & e^{x_5} & -x_3e^{x_5} & x_2 & x_1 & 0 \\ 0 & 0 & e^{x_5} & 0 & x_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & x_6 & 1 \end{bmatrix}, b \neq 0, T_2 = \begin{bmatrix} e^{x_5} & -x_3e^{x_5} & x_2 & x_5e^{x_5} & x_1 & 0 \\ 0 & e^{x_5} & 0 & 0 & x_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{x_5} & x_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & x_6 & 1 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} e^{x_5} & x_5e^{x_5} & \frac{1}{2}(2x_2 + x_5^2)e^{x_5} & -x_3 & x_1 & 0 \\ 0 & e^{x_5} & x_5e^{x_5} & 0 & x_4 & 0 \\ 0 & 0 & e^{x_5} & 0 & x_3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & x_6 & 1 \end{bmatrix}, T_4 = \begin{bmatrix} e^{x_5} & -x_3 & 0 & x_2e^{x_5} & x_1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{x_5} & x_5e^{x_5} & x_4 & 0 \\ 0 & 0 & 0 & e^{x_5} & x_3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & x_6 & 1 \end{bmatrix}$$

$$T_5 = \begin{bmatrix} e^{x_5} & x_4e^{x_5} & \frac{1}{2}(2ax_5 + x_4^2)e^{x_5} & x_2 & x_1 & 0 \\ 0 & e^{x_5} & x_4e^{x_5} & x_3 & x_2 & 0 \\ 0 & 0 & e^{x_5} & 0 & x_3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & x_6 & 1 \end{bmatrix}, a \in \mathbb{R},$$

$$T_6 = \begin{bmatrix} e^{x_5} & 0 & 0 & x_2 & 0 & 0 \\ 0 & e^{x_4} & 0 & 0 & x_1 & 0 \\ 0 & 0 & e^{ax_5+bx_4} & ax_3 & bx_3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & x_6 & 1 \end{bmatrix}, a^2 + b^2 \neq 0, \quad T_7 = \begin{bmatrix} e^{ax_4+x_5} & 0 & 0 & x_4 & x_1 & 0 \\ 0 & e^{x_4} & x_5e^{x_4} & x_2 & x_3 & 0 \\ 0 & 0 & e^{x_4} & x_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & x_6 & 1 \end{bmatrix}, a \in \mathbb{R},$$

$$T_8 = \begin{bmatrix} e^{ax_5+bx_4} & 0 & 0 & bx_1 & ax_1 & 0 \\ 0 & e^{x_4}cos(x_5) & e^{x_4}sin(x_5) & -x_2 & x_3 & 0 \\ 0 & -e^{x_4}sin(x_5) & e^{x_4}cos(x_5) & x_3 & x_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & x_6 & 1 \end{bmatrix}, a^2 + b^2 \neq 0$$

$$T_9 = \begin{bmatrix} 1 & x_4 & \frac{1}{2}x_4^2 & x_1 & 0 & 0 \\ 0 & 1 & x_4 & x_2 & 0 & 0 \\ 0 & 0 & 1 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{x_6} & x_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, T_{10} = \begin{bmatrix} e^{x_4} & 0 & 0 & x_1 & 0 & 0 \\ 0 & 1 & x_4 & x_2 & 0 & 0 \\ 0 & 0 & 1 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{x_6} & x_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{11}=\begin{bmatrix}1 & x_2 & x_1 & 0 & 0 & 0\\ 0 & 1 & x_3 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & e^{x_6} & x_6e^{x_6} & x_4\\ 0 & 0 & 0 & 0 & e^{x_6} & x_5\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}, T_{12}=\begin{bmatrix}1 & x_2 & x_1 & 0 & 0 & 0\\ 0 & 1 & x_3 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & e^{x_6} & 0 & x_4\\ 0 & 0 & 0 & 0 & e^{x_6} & x_5\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

$$T_{13}=\begin{bmatrix}1 & x_2 & x_1 & 0 & 0 & 0\\ 0 & 1 & x_3 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & e^{x_6} & 0 & x_4\\ 0 & 0 & 0 & 0 & e^{hx_6} & x_5\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}, -1\le h<1, h\neq 0$$

$$T_{14}=\begin{bmatrix}1 & x_2 & x_1 & 0 & 0 & 0\\ 0 & 1 & x_3 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & e^{px_6}cos(x_6) & e^{px_6}sin(x_6) & x_4\\ 0 & 0 & 0 & -e^{px_6}sin(x_6) & e^{px_6}cos(x_6) & x_5\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}, p\ge 0$$

$$T_{15}=\begin{bmatrix}e^{x_2} & x_1 & x_3 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & e^{x_6} & x_6e^{x_6} & x_4\\ 0 & 0 & 0 & 0 & e^{x_6} & x_5\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}, T_{16}=\begin{bmatrix}e^{x_2} & x_1 & x_3 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & e^{x_6} & 0 & x_4\\ 0 & 0 & 0 & 0 & e^{x_6} & x_5\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

$$T_{17}=\begin{bmatrix}e^{x_2} & x_1 & x_3 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & e^{x_6} & 0 & x_4\\ 0 & 0 & 0 & 0 & e^{hx_6} & x_5\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}, -1\le h<1, h\neq 0$$

$$T_{18}=\begin{bmatrix}e^{x_2} & x_1 & x_3 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & e^{px_6}cos(x_6) & e^{px_6}sin(x_6) & x_4\\ 0 & 0 & 0 & -e^{px_6}sin(x_6) & e^{px_6}cos(x_6) & x_5\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}, p\ge 0.$$

3.3 The matrix representation with a 2-dimensional centre

In this instance, the set of 3-dimensiona $Mult(L)$ topological loops L (cf. [1], §5) form eight families of Lie groups. Using [24] §4, the appropriate matrix Lie group $G_i, i=1,\dots,8$, in simple form is defined as follows.

$$\mathcal{M}_1=\begin{bmatrix}e^{ax_4} & 0 & 0 & x_1 & 0 & 0\\ 0 & e^{x_4} & x_4e^{x_4} & x_2 & 0 & 0\\ 0 & 0 & e^{x_4} & x_3 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & x_5 & 1 & 0\\ 0 & 0 & 0 & x_6 & 0 & 1\end{bmatrix}, a\neq 0, \mathcal{M}_2=\begin{bmatrix}e^{x_4} & x_4e^{x_4} & \frac{1}{2}x_4^2e^{x_4} & x_1 & 0 & 0\\ 0 & e^{x_4} & x_4e^{x_4} & x_2 & 0 & 0\\ 0 & 0 & e^{x_4} & x_3 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & x_5 & 1 & 0\\ 0 & 0 & 0 & x_6 & 0 & 1\end{bmatrix}$$

$$\mathcal{M}_3=\begin{bmatrix}e^{x_4} & 0 & 0 & x_1 & 0 & 0\\ 0 & e^{ax_4} & 0 & x_2 & 0 & 0\\ 0 & 0 & e^{bx_4} & x_3 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & x_5 & 1 & 0\\ 0 & 0 & 0 & x_6 & 0 & 1\end{bmatrix}, -1\le a\le b\le 1, ab\neq 0$$

$$\mathcal{M}_4=\begin{bmatrix} e^{ax_4} & 0 & 0 & x_1 & 0 & 0\\ 0 & e^{bx_4}cos(x_4) & e^{bx_4}sin(x_4) & x_2 & 0 & 0\\ 0 & -e^{bx_4}sin(x_4) & e^{bx_4}cos(x_4) & x_3 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & x_5 & 1 & 0\\ 0 & 0 & 0 & x_6 & 0 & 1\end{bmatrix}, a\neq 0, b\geq 0,$$

$$\mathcal{M}_5=\begin{bmatrix} e^{cx_5} & 0 & 0 & 0 & x_4 & 0\\ 0 & e^{x_5} & 0 & 0 & x_3 & 0\\ 0 & 0 & 1 & x_5 & x_1 & 0\\ 0 & 0 & 0 & 1 & x_2 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & x_6 & 1\end{bmatrix}, 0<|c|<|, \mathcal{M}_6=\begin{bmatrix} e^{x_5} & 0 & 0 & 0 & x_4 & 0\\ 0 & 1 & x_5 & \frac{1}{2}x_5^2 & x_1 & 0\\ 0 & 0 & 1 & x_5 & x_2 & 0\\ 0 & 0 & 0 & 1 & x_3 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & x_6 & 1\end{bmatrix}$$

$$\mathcal{M}_7=\begin{bmatrix} 1 & x_2 & 0 & 0 & x_1 & 0\\ 0 & 1 & 0 & 0 & x_5 & 0\\ 0 & 0 & e^{px_5}cos(x_5) & e^{px_5}sin(x_5) & x_4 & 0\\ 0 & 0 & -e^{px_5}sin(x_5) & e^{px_5}cos(x_5) & x_3 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & x_6 & 1\end{bmatrix}, p\neq 0, \mathcal{M}_8=\begin{bmatrix} e^{x_5} & x_5e^{x_5} & 0 & 0 & x_1 & 0\\ 0 & e^{x_5} & 0 & 0 & x_2 & 0\\ 0 & 0 & 1 & x_5 & x_3 & 0\\ 0 & 0 & 0 & 1 & x_4 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & x_6 & 1\end{bmatrix}$$

3.3.1 Example: - In case 3.3. Let $g_2=\mathbb{R}^2\oplus g_{4,4}=<f_1,f_2>\oplus<e_1,e_2,e_3,e_4>$ be a six-dimensional decomposable solvable Lie algebra with 2-dimensional centre defined the Lie bracket in [1] as follows $[e_1,e_4]=e_1$, $[e_2,e_4]=e_1+e_2$, $[e_3,e_4]=e_2+e_3$ and $\mathcal{K}_2=<e_1+f_1, e_2+a_2f_1, e_3+a_3f_1>$, $a_2,a_3\in\mathbb{R}$. We show that the Lie group G_2 allow continuous left transversals $\mathcal{S}$ and $\mathcal{T}$ to the subgroup $\mathcal{K}_2$ such that for all $\mathfrak{s}\in\mathcal{S}$ and $\mathfrak{t}\in\mathcal{T}$ one has $\mathfrak{s}^{-1}\mathfrak{t}^{-1}\mathfrak{s}\,\mathfrak{t}\in\mathcal{K}_2$ and the set $\mathcal{S}\cup\mathcal{T}$ generate G_2.

Proof: - We will choose two transversals $\mathcal{S}$ and $\mathcal{T}$ to the group $\mathcal{K}_2$ in G_2as

$$\mathcal{S}=\{g(h_1(u,v,w),h_2(u,v,w),h_3(u,v,w),u,v,w),u,v,w\in\mathbb{R}\},$$
$$\mathcal{T}=\{g(g_1(x,y,z),g_2(x,y,z),g_3(x,y,z),x,y,z),x,y,z\in\mathbb{R}\},$$

Where $h_i(u,v,w):\mathbb{R}^3\to\mathbb{R}$ and $g_i(x,y,z):\mathbb{R}^3\to\mathbb{R}$, $i=1,2,3$, be continuous functions such that $h_i(0,0,0)=g_i(0,0,0)=0$, by using the $\mathcal{M}_2$ in case 3.3. we get the product $\mathfrak{s}^{-1}\mathfrak{t}^{-1}\mathfrak{s}\,\mathfrak{t}\in\mathcal{K}_2$ if and only if the equations

$$e^{-x}(1-e^{-u})\left[g_1(x,y,z)+a_2g_2(x,y,z)+a_3g_3(x,y,z)-a_2xg_3(x,y,z)-xg_2(x,y,z)+\tfrac{1}{2}x^2g_3(x,y,z)\right]-e^{-u}(1-e^{-x})\left[h_1(u,v,w)+a_2h_2(u,v,w)+a_3h_3(u,v,w)-uh_2(u,v,w)+\tfrac{1}{2}u^2h_3(u,v,w)-a_2uh_3(u,v,w)\right]+e^{-u-x}[ug_2(x,y,z)-\tfrac{1}{2}u^2g_3(x,y,z)+uxh_3(u,v,w)-uxg_3(x,y,z)-a_2xh_3(u,v,w)+a_2ug_3(x,y,z)+\tfrac{1}{2}x^2h_3(u,v,w)-xh_2(u,v,w)=0 \quad \ldots\ldots\ldots\ldots (1).$$

Equation (1) holds precisely if the $\mathcal{K}_2$-connected transversals in G_2 are the set

$$\mathcal{S}=\left\{g\left(e^u-1-u^3+\frac{3}{2}a_2u^2+u(a_3-a_2^2), a_2u-\frac{3}{2}u^2,-u,u,v,w\right),u,v,w\in\mathbb{R}\right\},$$
$$\mathcal{T}=\left\{g\left(e^x-1-x^3+\tfrac{3}{2}a_2x^2+x(a_2^2-a_3),\tfrac{3}{2}x^2-a_2x,x,x,y,z\right),x,y,z\in\mathbb{R}\right\},a_2,a_3\in\mathbb{R}.$$

In this case, the set $\mathcal{S}\cup\mathcal{T}$ generate the group G_2. Hence, by Proposition 2.1. $G_2\cong Mult(L)$.

4. Conclusion

The explanation regarding finding linear representations for six-dimensional decomposable solvable Lie algebras which are multiplication groups of 3-dimensional topological loops L with L satisfy two important properties connected simply connected, is done. This work provides a novel framework for deriving explicit matrix representations of the decomposable solvable Lie algebras. The next idea is to consider the 7-dimensional solvable Lie algebras as the Lie algebra of the multiplication group of a 3-dimensional connected topological loop. Those Lie algebras are not fully classified till now. However, in some special cases, the 7-dimensional decomposable solvable Lie algebras are determined.

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Conflicts of Interest

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التمثيل المصفوفي لجبر لي المحلول القابل للتفكيك سداسي الأبعاد

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الخلاصة:

تُوسّع هذه الورقة تصنيف اللوب الطوبولوجية المتصلة ثلاثية الأبعادL ، حيث تكون مجموعة الضرب فيها عبارة عن زمرة لي سداسية الأبعاد قابلة للتحلل والحل. بناءً على النتائج المهمة الواردة في [1]، والتي أثبتت انعدام القوى المركزية من الدرجة الثانية لهذه الوب، نستنتج وبشكل صريح تمثيلات مصفوفيه لجبر لي المرتبط بها. تُعد هذه التمثيلات بالغة الأهمية لإكمال التصنيف، إذ تُسهّل التحقق من التوافق مع الشروط العملية المعرفة على جبر لي. نحدد 26 عائلة مميزة من زمر لي المصفوفية، بما في ذلك 18 مجموعة تتميز بأحادية المركز و8 مجموعات ثنائية المركز. بالإضافة إلى ذلك، نُوضح التوافق بين هذه المجموعات وجبر لي المُشتق منها. تُعالج نتائجنا فجوة قائمة في الأدبيات من خلال توفير إطار منهجي لدراسة العلاقات المتبادلة بين الوب الطوبولوجية وزمر لي المُناظرة لها.