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Complex dynamics of a family Cubic-Logistic map

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Abstract

We introduce a new family of the Logistic map, namely the cubic logistic maps $\boldsymbol{\mathcal{L}} = \{\boldsymbol{L_\lambda(x) = \lambda x^2(1-x)}: \boldsymbol{\lambda > 0, x} \in \mathbb{R}\}$. In this work we study the complex dynamics of this family i.e. $\boldsymbol{\mathcal{L}} = \{\boldsymbol{L_\lambda(z) = \lambda z^2(1-z)}: \boldsymbol{\lambda > 0, z} \in \mathbb{C}\}$. That is, we study the Fatou and Julia sets of this map. In fact, we give a whole description for these sets. These two new types of logistics maps can address some of life's problems as shown in the introduction. We prove for any $\boldsymbol{\lambda} \in \mathbb{R}$, $\boldsymbol{L_\lambda} \in \boldsymbol{\mathcal{L}}$ is preservers $\mathbb{R}$, critically finite, maps the negative x-axis into positive real line and has no any complex periodic point. Fatou set of these maps has no Siegel disk, Baker domain, and has no Wandering domain so they consist of parabolic domains and basins of attraction. Finally, we use escaping algorithm to construct the Fatou and Julia sets of our maps for various values of $\boldsymbol{\lambda}$.

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1. Introduction

The complex logistic map is an extension of the classic logistic map, which has a rich history in dynamical systems and chaos theory. The original logistic equation was first introduced by Pierre François Verhulst in 1838 to model population growth. This equation became a cornerstone in the study of nonlinear dynamics due to its simple form yet complex behavior, particularly when applied to chaotic systems. For further reading, these sources provide deeper insights see [2],[3] and [4]. The standard form of the logistic map is $f(x) = \lambda x(1-x)$. where $\boldsymbol{\lambda}$ is the growth rate, and $\boldsymbol{x}$ represents the population at a given time.

In addition, some studies focused on simulating the logistic map by modifying the map, In [5] the authors studied the First Modified Logistic (FML) Map and (Second Modified Logistic (SML) Map)

$$x_{n+1} = FML(r, x_n) = \begin{cases} r.x_n.(1-x_n), & x_n < 0.5 \\ r.(x_n - 0.5).(x_n - 1.5) + r/4, & x_n \geq 0.5 \end{cases}$$

, where $\boldsymbol{x_n} \in [0,1]$, $r \in (0,4]$

$$x_{n+1} = SML(r, x_n) = \begin{cases} r.x_n.(1-x_n), & x_n < 0.5 \\ r.x_n.(x_n - 1) + r/4, & x_n \geq 0.5 \end{cases}$$

,where $\boldsymbol{n}$ is a time index , x_0 is the initial value , $x_n \in [0,1]$ and the control parameter $r \in [0,1]$. In [6] the authors studied $x(t+1) = (1-\varepsilon)f(x(t))$, $f(x) = \mu x(1-x^2)$, $y(t+1) = o(x(t+1))$, $x \in (-1,1)$ where $x(t)$ is the internal state of the neuron at time t, $y(t)$ is the output of the neuron at time t. μ and ε are bifurcation parameter. In [7] the authors studied $x_{n+1} = a(bx^3{}_n - tx_n), n \in \mathbb{N}, x_n \in \mathbb{R}$, where parameter $a \in \{-1,1\}, b \in (1,10)$ and $t \in (1,3)$. We introduce in [1] a new family of logistic $\lambda x^2(1-x)$, namely cubic logistic map. In this work we study the complex dynamics of our family, i.e. we study the family of maps $L_\lambda(z) = \lambda z^2(1-z)$, $\lambda > 0$, $z \in \mathbb{C}$. We describe the Fatou ($F(L_\lambda)$) and Julia ($J(L_\lambda)$) sets completely $\forall\, \lambda \in R$. We show that if $\lambda > 0$ and $L_\lambda(z) = \lambda z^2(1-z)$. Then

1- for $0 < \lambda < 4$, $F(L_\lambda) = B(0)$

2- For $\lambda = 4$, $F(L_\lambda) = B(p) \cup P(x_1)$, where $B(p)$ is the basin of attraction of the attracting fixed point $p = 0$ and the parabolic domain of the indifferent fixed point $x_1 = \frac{1}{2}$.

3- For $4 < \lambda < 5.333$, $F(L_\lambda) = B(0) \cup B(a_\lambda)$.

4- For $\lambda = 5.333$, $F(L_\lambda) = B(0) \cup P(\frac{3}{4})$.

5- For $\lambda > 5.333$, $F(L_\lambda) = B(0)$, i.e. $J(L_\lambda) = \mathbb{C} - B(0)$.

So, the Fatou set of our maps cannot be empty and its always ($\forall\, \lambda > 0$) contains the basin of attraction of the attracting fixed point $z = 0$. Also, we show that the Fatou sets cannot contains any Baker domain, Herman ring, Wandering domain and Siegel disk. Thus, they contain only Basins of the attraction and parabolic domains.

Basic Definitions and Theorems: In this section we remember some basic definitions and some theorems which we need them in our work.

2. PRELIMINARIES

Definition 2.1

Let X be a metric space and let $h: X \to X$ be a function, let $z \in X$. Then the point z is called a fixed point of the function h if $h(z) = z$.

Definition (2.2): [9]

Let X be a metric space and let $h: X \to X$ be a function, let $x \in X$. Then the point x is called a "periodic point" of the function h if there exist $n \in N$ such that $h^n(x) = x$. The smallest number m satisfies that $h^m(x) = x$ is called the period of x. Note that the fixed point x is a periodic point of the period one.

Definition (2.3): [9]

For a periodic point x with period n, the multiplier of x is $\left|\left(f^{n\prime}(x)\right)\right|$.

Definition (2.4): [8]

For $f: X \to X$, Orbit of $z \in X$ is defined as the set of points $O\ (z) = \{z, f(z), f^2(z), \dots\}$.

Definition (2.5):[10]

A transcendental function is an analytic function that does not satisfy a polynomial equation.

Definition (2.6): [9]

Let $\mathbb{C}$ be a metric space and let $h: \mathbb{C} \to \mathbb{C}$ be an analytic function, let $z \in \mathbb{C}$. Then the point z is called indifferent point if $\left|\acute{h}(z)\right| = 1$.

Definition (2.7): [9]

Let $f(z)$ be an analytic function in $\mathbb{C}$. A point z_0 is called an attracting point (repelling point) for the function $f(z)$ if $|f'(z_0)| < 1$ ($|f'(z_0)| > 1$).

Definition (2.8):[11]

Let f be analytic map.

1- A point $w \in \mathbb{C}$ is a critical point for f if $\acute{f}$(w)=0.

2- The value of $f(w)$ is called a critical value of f.

3- If we denote the extended complex plane, i.e., the complex plane with the point at infinity, by $\hat{\mathbb{C}}$, then the point v is said to be asymptotic value for f if there is a continuous path $\gamma: (0, \infty) \longrightarrow \hat{\mathbb{C}}$, satisfying $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} f(\gamma(t)) = v$.

4- A singular value of f is defined to be either critical or asymptotic value of f.

5- The function which has only finitely many singular values is called critically finite , otherwise it is called non critically finite.

Definition (2.9): [4]

Let $F = \{f_n: n \in N\}$ be a family of complex maps defined on an open set U of $\hat{\mathbb{C}}$. F is said to be normal family on U if every infinite sequence of F contains a subsequence which either:

1- Converges uniformly on compact sets of U, or

2- Converges uniformly to ∞ on U.

Definition (2.10):[12]

The Fatou set of a complex function f, denoted by $F(f)$, is defined as $F(f) = \{z \in \hat{C}$: the sequence $< f_n >$ is normal in some neighborhood of $z\}$. Julia set, denoted by $J(f)$, of f is the complement of $F(f)$, that is $J(f) = \hat{C}/F(f)$.

Definition (2.11):[13]

An entire function is a function that is analytic at each point in the plane.

Since the derivative of a polynomial exists everywhere, it follows that every polynomial is an entire function.

Definition (2.12): [14]

$I(f) = \{z \in \mathbb{C}: f^n(z) \to \infty$ as $n \to \infty\}$. $I(f)$ is called the escaping points set of f.

Theorem (2.13): [15]

Let f be a complex function. Let U be a periodic component, of period p, in the Fatou set of f. Then one of the following possibilities is true:

1) U contains an attracting periodic point z_0 of period p. Then for $z \in U$, $f^{np}(z) \to z_0$ as $n \to \infty$ and U is called the basin of attracting of z_0.

2) ∂U contains a periodic point z_0 , $f^{np}(z) \to z_0$ as $n \to \infty$ for $z \in U$ and $(f^{p\prime}(z_0)) = 1$. In this case U is called parabolic domain.

3) There is an analytic homomorphism $\emptyset: U \to D$, where D is the unit disk, such that $\emptyset \circ f^p \circ$

$\emptyset^{-1}(z) = e^{2\pi i\alpha}z$, for some $\alpha \in R/Q$. In this case U is called a Siegel disk.

4) There exists an analytic homomorphism $\emptyset: U \to A$, where $A = \{z \in \mathbb{C}: 1 < |z| < r\}$, $r > 1$, such that $\emptyset \circ f^p \circ \emptyset^{-1}(z) = e^{2\pi i\alpha}z$, for some $\alpha \in R/Q$. In this case U is called Herman ring.

5) There exists $z_0 \in U$ such that $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$ but $f^{np}(z_0)$ is not defined. In this case, U is called Baker domain.

If U is not periodic then U is called wandering domain, i.e. if U is wandering domain then $U^m \neq U^n$ for all $m \neq n$.

Theorem (2.14): [11]

Let f be a transcendental meromorphic function and let $D = \{U_0, U_1, \ldots, U_{n-1}\}$ be a n-periodic cycle of components of $F(f)$. Then:

1- If D is a cycle of attracting basins or parabolic domains then some U_k , $k \in \{1,2, \ldots, n-1\}$, must intersect the of singular values of f, which is denoted by $S(f)$.

2- If D is a cycle of Siegel disks or Herman rings then ∂U_k is a proper subset of the closure of the set of forward orbits of $S(f)$ for each $k \in \{1,2, \ldots, n-1\}$.

Lemma (2.15): [16]

The Fatou set of an entire function does not have Herman rings.

Lemma (2.16): [16]

If f is a critically finite function then the Fatou set of f has no wandering domain.

Lemma (2.17): [16]

If f is a critically finite function then the Fatou set of f has no Baker domain.

Theorem (2.18): [14]

Let B be the class of meromorphic function having bounded singular values. for $f \in B, J(f)$ =the closure of the set $I(f)$.

In [1] we proved the following series of theorems for $\mathcal{L} = \{L_\lambda(x) = \lambda x^2(1-x): \lambda \in \mathbb{R}, x \in \mathbb{R}\}$.

Theorem (2.19): Let $L_\lambda \in \mathcal{L} = \{L_\lambda(x) = \lambda x^2(1-x): \lambda > 0, x \in \mathbb{R}\}$, For $\lambda = 4$, then:

1- $L_\lambda^n(x) \to 0$ for $x \in (-x_1, x_1)$.

2- $L_\lambda^n(x) \to x_1$ for $x \in (-1, -x_1) \cup (x_1, 1)$.

3- $L_\lambda^n(x) \to \infty$ for $x \in (-\infty, -1) \cup (1, \infty)$.

where $x_1 = \frac{1}{2}$ is the indifferent fixed point for F

Theorem (2.20): Let $L_\lambda \in \mathcal{L} = \{L_\lambda(x) = \lambda x^2(1-x): \lambda > 0, x \in \mathbb{R}\}$, For $4 < \lambda < 5.333$, the dynamics of $L_\lambda(x)$ is as follows:

1- For $x \in (-\infty, -r_\lambda)$, $L_\lambda^n(x) \to \infty$.

2- For $x \in (-r_\lambda, r_\lambda)$, $L^n(x) \to 0$.

3- For $x \in (r_\lambda, 1)$, $L^n(x) \to a_\lambda$.

4- For $x \in (1, \infty)$, $L^n(x) \to \infty$.

Where r_λ is the repelling fixed point fo$L_\lambda(x)$, and a_λ is attracting

Theorem (2.21): Let $L_\lambda \in \mathcal{L} = \{L_\lambda(x) = \lambda x^2(1-x): \lambda > 0, x \in \mathbb{R}\}$, For $\lambda \approx 5.333$, we have two fixed points $x^* = \frac{1}{4}, x_2 = \frac{3}{4}$. The point $x^* = \frac{1}{4}$ is repelling and $x = \frac{3}{4}$ is indifferent and the dynamics of $L_\lambda(x)$ as follows:

1- For $x \in (-\frac{1}{4}, \frac{1}{4})$, $L^n(x) \to 0$.

2- For $x \in (\frac{1}{4}, \frac{3}{4}) \cup (\frac{3}{4}, 1)$, $L^n(x) \to \frac{3}{4}$.

3- For $x \in (-\infty, -\frac{1}{4}) \cup (1, \infty)$, $L^n(x) \to \infty$.

Theorem (2.22): Let $L_\lambda \in \mathcal{L} = \{L_\lambda(x) = \lambda x^2(1-x): \lambda > 0, x \in \mathbb{R}\}$, For $\lambda > 5.333$, Then

1- $L^n{}_\lambda(x) \to 0$ for $x \in (-r_\lambda^*, r_\lambda^*) \cup (r_\lambda^{**}, 1)$.

2- $L_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (-\infty, -r_\lambda^*) \cup (r_\lambda^*, r_\lambda^{**}) \cup (1, \infty)$.

Where r_λ^* , r_λ^{**} are repelling fixed points for $L_\lambda(x)$.

It is important remember that $x = 0$ is attracting fixed point for $L_\lambda(x)$, $\forall\ \lambda \in \mathbb{R}$.[1]

3. Main results

In this section we study the complex dynamic of

$\mathcal{L} = \{L_\lambda(z) = \lambda z^2(1-z): \lambda \in \mathbb{R}, z \in \mathbb{C}\}$, i.e. $F(L_\lambda)$ and $J(L_\lambda)$, $\lambda > 0$.

Firstly, we prove the following simple lemmas we need them in our work.

Lemma (3.1):

For any $\lambda \in \mathbb{R}$, $L_\lambda \in \mathcal{L} = \{L_\lambda(z) = \lambda z^2(1-z): \lambda > 0, z \in \mathbb{C}\}$ preservers $\mathbb{R}$.

Proof: $L_\lambda(z) = \lambda z^2(1-z) = \lambda(x+yi)^2(1-x-yi)$

$= \lambda(x^2 - y^2 + 2xyi)(1-x-yi)$

$= \lambda(x^2 - x^3 - x^2yi - y^2 - y^2x + y^3i + 2xyi - 2x^2yi + 2xy^2)$

$= \lambda(x^2 - x^3 - y^2 - y^2x + 2xy^2) + \lambda i(-x^2y + y^3 + 2xy - 2x^2y)$

For $z \in \mathbb{R}$, i.e $z = x$ and $y = 0$

$L_\lambda(z) = \lambda(x^2 - x^3) = \lambda x^2(1-x)$

Thus L preservers $\mathbb{R}$.

Lemma (3.2):

For $L_\lambda \in \mathcal{L} = \{L_\lambda(z) = \lambda z^2(1-z): \lambda > 0, z \in \mathbb{C}\}$ is critically finite.

Proof:

L_λ is a polynomial. So it has no asymptotic value.

On the other hand, the critical points are 0 and $2/3$ only.

This means that the singular values of L_λ are $s_1 = 0$, $s_2 = 0.148\lambda$. So L_λ is critically finite.

Lemma (3.3):

Let $L_\lambda \in \mathcal{L} = \{L_\lambda(z) = \lambda z^2(1-z): \lambda > 0, z \in \mathbb{C}\}$ maps $(1, \infty)$ into $(-\infty, 0)$ and map the negative real line into positive real line.

Proof:

For $\lambda > 0$, $x > 1$, it is clear that $\lambda x^3 > \lambda x^2$, so $\lambda x^2(1-x) < 0$, i.e. $L_\lambda(1, \infty) \subseteq (-\infty, 0)$.

And for $x < 0$, $\lambda x^2 > \lambda x^3$ ($\lambda > 0$) and $L_\lambda(0) = 0$. So $L_\lambda(x) = \lambda x^2 - \lambda x^3 > 0$ and $L_\lambda(-\infty, 0) \subseteq (0, \infty)$, i.e. the negative real line mapped by L into positive real line.

Lemma (3.4):

Let $L_\lambda \in \mathcal{L} = \{L_\lambda(z) = \lambda z^2(1-z): \lambda > 0, z \in \mathbb{C}\}$ has no complex periodic point.

Proof:

Assume z_0 is a periodic point and z_0 not real. By Theorem (2.14) there exists a singular value ,i.e. either 0 or 0.148λ, such that $\lim_{n\to\infty}\left({L^{nk}}_\lambda(s_i)\right) = z_0$, $i = 1$ or 2 where , $s_1 = 0$ or $s_2 = 0.148\lambda$ and k is a period of z_0. But, by lemma (3.1) the map L_λ is preserves $\mathbb{R}$, i.e. ${L^{nk}}_\lambda(s_i) \in \mathbb{R}$ which is a contradiction with z_0 is a complex number. Thus L_λ has no any periodic complex point. □

Now, we start to study the complex dynamics of the family $\mathcal{L} = \{L_\lambda(z) = \lambda z^2(1-z): \lambda > 0, z \in \mathbb{C}\}$. That is, we describe the Julia and Fatou sets of $L_\lambda \in \mathcal{L}, \forall\, \lambda > 0$.

Proposition (3.5):

For any $\lambda > 0$, the Fatou set of $L_\lambda(z)$, $L_\lambda(z) \in \mathcal{L} = \{L_\lambda(z) = \lambda z^2(1-z): \lambda > 0, z \in \mathbb{C}\}$ has no Baker domain, wandering domain, and has no Herman ring.

Proof:

Since L_λ is entire and critically finite (lemma (3.2)) . Then by lemma (2.17) and lemma(2.16) , $F(L_\lambda)$ can not have neither Baker domain nor wandering domain. Also, $F(L_\lambda)$ has no Herman ring by lemma(2.15).

Proposition (3.6):

For any $\lambda > 0$, $L_\lambda(z) \in \mathcal{L} = \{L_\lambda(z) = \lambda z^2(1-z): z \in \mathbb{C}\}$. The Fatou set of L_λ, $F(L_\lambda)$ contains a basin of attraction of the attracting fixed point $z = 0$, $B_\lambda(0)$.

Proof:

We proved in [1], $\forall\, \lambda \in \mathbb{R}$, the point $z = 0$ is a fixed point for L_λ, and it is attracting. In the sequence of theorems (2.19) to (2.22) we showed that for certain intervals are asymptotic to $z = 0$. Hence these intervals are contained in the basins of attraction of 0.

The following theorems study the complex dynamic of our family for $\lambda > 0, z \in \mathbb{C}$:

let $\lambda > 0$, $\mathcal{L} = \{L_\lambda(z) = \lambda z^2(1-z): \lambda > 0, z \in \mathbb{C}\}$, $x_1 = \frac{1}{2}$, $x_2 = \frac{3}{4}$, a_λ , r_λ , r_λ^* and r_λ^{**} as given in theorems (2.19) to (2.22) . The following theorems describe the Fatou and Julia sets of L_λ.

Theorems (3.7):

For $\lambda = 4$, the Fatuo set contains the intervals $(-1, -x_1)$,$(-x_1, x_1)$ and $(x_1, 1)$. And the intervals $(-\infty, -1)$ and $(1, \infty)$ are contained in Julia set, of L_λ.

Proof :

1) By theorem (2.19) , we proved that ${L^n}_\lambda(x) \to 0$ $\forall\, x \in (-x_1, x_1)$. Thus the sequence $\{{f^n}_\lambda(x)\}$ is a normal family for $x \in (-x_1, x_1)$. That is $x \in F(L_\lambda)$, $\forall\, x$ satisfies the convergent above. By the same theorem , we proved that ${L^n}_\lambda(x) \to x_1$ for $x \in (-1, -x_1) \cup (x_1, 1)$. So $\{L^n(x)\}$ is also a normal sequence and hence the intervals $(-1, -x_1) \cup (x_1, 1)$ are contained in $F(L_\lambda)$.
2) By theorem (2.18), $\forall\, z \in \mathbb{C}$, $f^n(z) \to \infty$ then $z \in J(f)$. Thus, for $\lambda = 4$, and by theorem (2.19), ${L^n}_\lambda(x) \to \infty$ for $x \in (-\infty, -1) \cup (1, \infty)$, so the intervals $(-\infty, -1)$ and $(1, \infty)$ are subsets of Julia set of L_λ i.e. $(-\infty, -1) \cup (1, \infty) \subseteq J(L_\lambda)$.

Theorem (3.8):

Let $\mathcal{L} = \{L_\lambda(z) = \lambda z^2(1-z): \lambda > 0, z \in \mathbb{C}\}$. For $4 < \lambda < 5.333$, then:

1- The intervals $(-\infty, -r_\lambda)$ and $(1, \infty) \subseteq J(L_\lambda)$

2- $(-r_\lambda, r_\lambda) \subseteq F(L_\lambda)$ and so is $(r_\lambda, 1)$.

Proof :

1) By theorem (2.20) , we proved that ${L^n}_\lambda(x) \to \infty$, for $x \in (-\infty, -r_\lambda)$. Then $x \in J(L_\lambda)$ and by the same theorem , ${L^n}_\lambda(x) \to \infty$, for $x \in (1, \infty)$. Hence the intervals $(-\infty, -r_\lambda)$ and $(1, \infty)$ are subset of Julia set of L_λ i.e. $(-\infty, -r_\lambda) \cup (1, \infty) \subseteq J(L_\lambda)$.
2) By theorem (2.20), we proved that ${L^n}_\lambda(x) \to 0$, for $x \in (-r_\lambda, r_\lambda)$. Thus the sequence $\{{L^n}_\lambda(x)\}$ is a normal family for $x \in (-r_\lambda, r_\lambda)$. That is $x \in F(L_\lambda)$, $\forall\, x$ satisfies the convergent above. By the same theorem , we proved that ${L^n}_\lambda(x) \to a_\lambda$ for $x \in (r_\lambda, 1)$. So $\{L^n(x)\}$ is also a normal sequence and hence

the intervals $(-r_\lambda, r_\lambda) \cup (r_\lambda, 1)$ are contained in $F(L_\lambda)$.

Theorem (3.9):

let $\mathcal{L} = \{L_\lambda(z) = \lambda z^2(1-z): \lambda > 0, z \in \mathbb{C}\}$, $\lambda = 5.333$ then :

1) $F(L_\lambda)$ contains $(-\frac{1}{4}, \frac{1}{4})$, $(\frac{1}{4}, \frac{3}{4})$ and $(\frac{3}{4}, 1)$.

2) $J(L_\lambda)$ contains $(-\infty, -\frac{1}{4})$ and $(1, \infty)$.

Proof :

1) By theorem (2.21) , we proved that $L^n{}_\lambda(x) \to 0 \ \forall x \in (-\frac{1}{4}, \frac{1}{4})$. Thus the sequence $\{L^n{}_\lambda(x)\}$ is a normal family for $x \in \left(-\frac{1}{4}, \frac{1}{4}\right)$. That is $x \in F(L_\lambda)$, $\forall x$ satisfies the convergent above. By the same theorem , we proved that $L^n{}_\lambda(x) \to \frac{3}{4}$ for $x \in (\frac{1}{4}, \frac{3}{4}) \cup (\frac{3}{4}, 1)$. So $\{L^n(x)\}$ is also a normal sequence and hence the intervals $(\frac{1}{4}, \frac{3}{4}) \cup (\frac{3}{4}, 1)$ are contained in $F(L_\lambda)$.

2) By theorem (2.21), we proved that, $L^n{}_\lambda(x) \to \infty$ for $x \in (-\infty, -\frac{1}{4}) \cup (1, \infty)$, then $x \in J(L_\lambda)$. So the intervals $(-\infty, -\frac{1}{4})$ and $(1, \infty)$ are subsets of Julia set of L_λ i.e. $(-\infty, -\frac{1}{4}) \cup (1, \infty) \subseteq J(L_\lambda)$.

Theorem (3.10):

Let $\mathcal{L} = \{L_\lambda(z) = \lambda z^2(1-z): \lambda > 0, z \in \mathbb{C}\}$. For $\lambda > 5.333$, then:

1- The intervals $(-r^*{}_\lambda, r^*{}_\lambda) \cup (r^{**}{}_\lambda, 1) \subseteq F(L_\lambda)$.

2- $(-\infty, -r^*{}_\lambda) \cup (r^*{}_\lambda, r^{**}{}_\lambda) \cup (1, \infty) \subseteq J(L_\lambda)$.

Proof:

1) By theorem (2.22) , we proved that $L^n{}_\lambda(x) \to 0$ for $x \in (-r^*_\lambda, r^*_\lambda) \cup (r^{**}_\lambda, 1)$. Thus the sequence $\{L^n{}_\lambda(x)\}$ is a normal family for $x \in (-r^*_\lambda, r^*_\lambda) \cup (r^{**}_\lambda, 1)$. That is $x \in F(L_\lambda)$. Hence the intervals $(-r^*_\lambda, r^*_\lambda)$ and $(r^{**}_\lambda, 1)$ are contained in $F(L_\lambda)$.

2) By theorem (2.22), we proved that, $L^n{}_\lambda(x) \to \infty$ as $n \to \infty$ for $x \in (-\infty, -r^*_\lambda) \cup (r^*_\lambda, r^{**}_\lambda) \cup (1, \infty)$, then $x \in J(L_\lambda)$. Hence the intervals $(-\infty, -r^*_\lambda)$, $(r^*_\lambda, r^{**}_\lambda)$ and $(1, \infty)$ are subsets of Julia set of L_λ i.e. $(-\infty, -r^*_\lambda) \cup (r^*_\lambda, r^{**}_\lambda) \cup (1, \infty) \subseteq J(L_\lambda)$.

The following theorem gives the full structures of Julia and Fatou set of L_λ, for different values of λ.

Theorem (3.11):

Let $\lambda > 0$, $L_\lambda(z) = \lambda z^2(1-z)$. Then:

1- For $\lambda = 4$, $F(L_\lambda) = B(p) \cup P(x_1)$, where $p = 0, x_1 = \frac{1}{2}$ are the attracting and indifferent periodic point for L_λ.

2- For $4 < \lambda < 5.333$, $F(L_\lambda) = B(0) \cup B(a_\lambda)$.

3- For $\lambda = 5.333$, $F(L_\lambda) = B(0) \cup P(x_1)$.

4- For $\lambda > 5.333$, $F(L_\lambda) = B(0)$.

Proof:

Let $L_\lambda(z) = \lambda z^2(1-z)$. First, note that, the singular values of L_λ are only the critical values $s_1 = 0$ and $s_2 = 0.148\lambda, \forall \lambda \in \mathbb{R}$ associated by the critical points $x_1 = 0$ and $x_2 = \frac{2}{3}$. Now , the orbits of $x_1 = 0$ and $x_2 = \frac{2}{3}$ are contained in the Fatou sets of $L_\lambda(z)$ as shown the previous theorems. Thus, by Theorem (2.14), $F(L_\lambda)$ has no any Siegel disks.

If we compain this result with proposition(3.5) we have:

1- For $\lambda = 4$, $L_\lambda(z)$ has only one attracting fixed point at $z = 0$ and only one indifferent fixed point x_1 , so $B(0)$ and $P(x_1)$ are contained in $F(L_\lambda)$. On the other hand, by Lemma(3.4), any periodic point must be real and hence contained in a one of Fatou set components of L_λ. Thus , $F(L_\lambda)$ dose not contains any components other than basin of $z = 0$ and parabolic domain of $z = x_1 = 1/2$.

2- When $4 < \lambda < 5.333$, $L_\lambda(z)$ has two attracting fixed points , the first at $z = 0$ and the second at $a_\lambda \in (0, 0.5)$ and $L_\lambda(z)$ has no any indifferent point so $F\big(L_\lambda(z)\big) = B(0) \cup B(a_\lambda)$.

3- When $\lambda = 5.333$, $L_\lambda(z)$ has two fixed points , the first at $z = 0$ and the second at x_1 which is indifferent. So $F\big(L_\lambda(z)\big) = B(0) \cup P(x_2)$.

4- When $\lambda > 5.333$, $L_\lambda(z)$ has one attracting fixed point at $z = 0$ and has no any indifferent point so $F\big(L_\lambda(z)\big) = B(0) = (-r^*_\lambda, r^{**}_\lambda) \cup (r^{**}_\lambda, 1)$.

Remark (3.12):

Note that Fatou set for cubic logistic family $\mathcal{L} = \{L_\lambda(z) = \lambda z^2(1-z): \lambda > 0, z \in \mathbb{C}\}$ cannot be empty , since $\forall \lambda \in \mathbb{R}$, $z = 0$ is attracting fixed point so $\emptyset \neq B(0) \subseteq F(L_\lambda)$, this mean. $J(L_\lambda) \neq \mathbb{C}$, i.e. we have no the exploding phenomenon in the Julia set of our family, but the (size) of Fatuo set of our family is "so small" (it is just some intervals in $\mathbb{R}$).

We end this work by construsting image of Julia and Fatou sets of our maps for various values of λ.

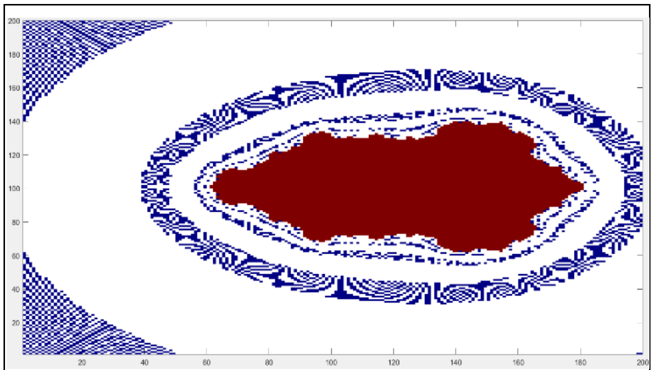

Figure 1. Julia and Fatou sets for $L_2(z)$

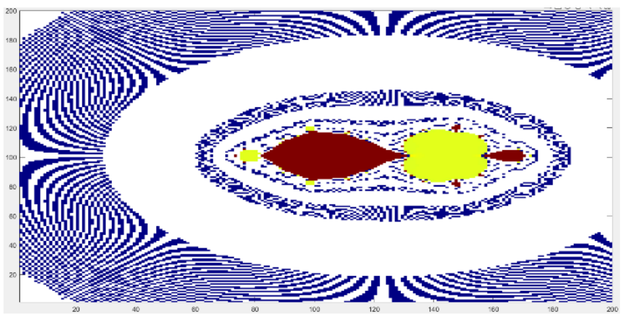

Figure 2. Julia and Fatou sets for $L_4(z)$

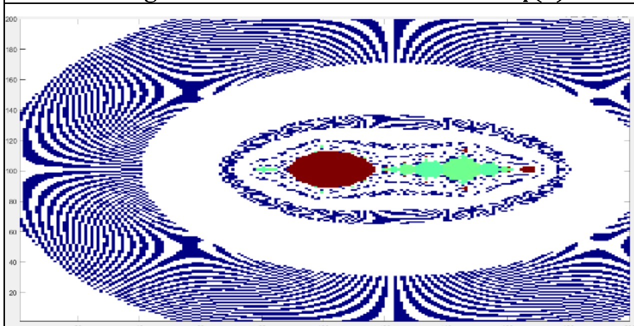

Figure 3. Julia and Fatou sets for $L_5(z)$

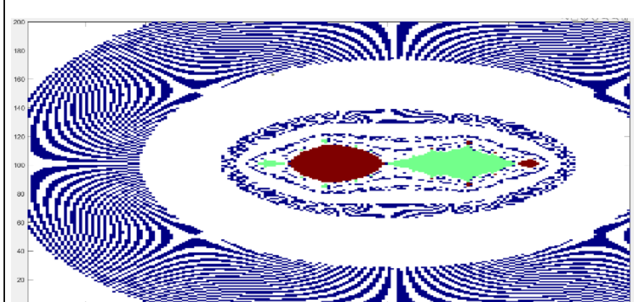

Figure 4. Julia and Fatou sets for $L_{5.333}(z)$

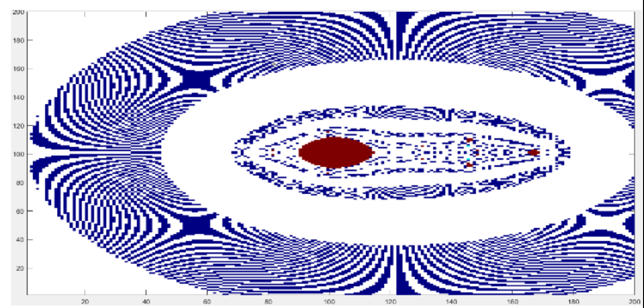

Figure 5. Julia and Fatou sets for $L_6(z)$

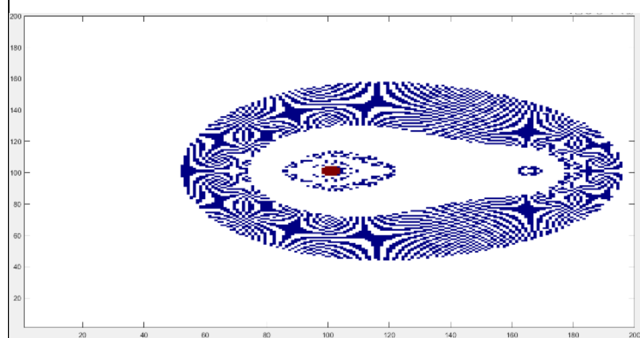

Figure 6. Julia and Fatou sets for $L_{20}(z)$

Note that the **Figures (1-8)** give the structures of Fatou and Julia sets

1. the brown space is the basin of zero
2. Green and yellow space are the parabolic domains.
3. The blue space is the Julia set.

Note that the “area” of Fatou set decreases as value of λ increases.

Conclusion

In this paper, we studied and analyzed the complex dynamics of a modified family of logistic maps, which we named the " cubic logistic map". Using a set of mathematical theories and analyses, we were able to determine the dynamic behavior of this family in different ranges of parameter values. We addressed the basic properties of fixed points and periodic points, and we give the whole structure of $J(L_\lambda)$, $F(L_\lambda)$ for various value of $\lambda > 0$ around the bifurcation values of λ, namely $\lambda = 4$ and $\lambda = 5.333$.

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Conflict of interest

None.

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