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A Novel Lotka-Volterra Model for Analyzing the Dynamic Relationship Between Financial Corruption and Society

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Abstract

In this paper, we introduce a new predator-prey system and prove the existence, uniqueness, and stability of the proposed system. The main tools used in the study are the Picard approximation iteration and the principles of Ulam stability. Our results contribute to the understanding of dynamical behaviors in predator-prey interactions and provide a theoretical foundation for further studies in ecological modeling.

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1. Introduction

The first system of equations modeling predator-prey systems was developed in 1925 by the American biophysicist Alfred Lotka, whose research aimed to demonstrate oscillating chemical processes, and the following year, the Italian mathematician Vito Volterra expanded on this idea by studying cyclical shifts in the populations of predatory fish and their prey in the Adriatic Sea during World War I. Since the war between Austria and Italy halted commercial fishing, the population of predatory fish increased compared to the years before the war, while the population of prey fish decreased [1], [7], [14], [13]. Later, this model was named after these two researchers and became known as the Lotka-Volterra model or the predator-prey model. Their work established the theoretical foundations of population biology and served as the basis for subsequent researchers' investigations into the dynamic behavior of biological populations [3], [10].

The Lotka-Volterra model is a dynamic predator-prey model that describes interactions among an arbitrary number of ecological competitors [8]. The Lotka-Volterra model has evolved into a versatile mathematical framework for understanding complex interactions. It was initially developed to describe the dynamics of interactions between two species, such as predators and prey, in biological systems. Essentially, the equations in the model illustrate how species populations change over time due to reciprocal interactions such as competition, predation, or symbiosis [4], [16], [11]. The Lotka-Volterra model is unique in that it neither converges nor diverges. Its long-term dynamical behavior is directly influenced by its initial conditions. Unlike ecosystems that often tend to reach a stable equilibrium state, the population size continuously fluctuates between a definite peak and a defined valley [3].

One of the most popular models for illustrating an ordinary non-linear control system is the predator-prey scenario. Numerous significant physical events are simulated using nonlinear differential equations in various scientific and technological fields. It is often impossible, or very difficult, to solve these problems analytically. However, there has been a notable increase in the use of analytical approximation methods in recent years to obtain reasonably accurate solutions [5], [12].

This study applies the Lotka-Volterra model to the interaction between financial corruption and the population in society. Combating corruption is crucial for the country's future growth and stability. This approach not only enhances the theoretical understanding of the financial dynamics of corruption but also provides policymakers and other stakeholders with valuable insights.

It is possible to modify the Lotka-Volterra model for use in social sciences to examine how a society's population dynamics and financial corruption interact. Financial corruption, a persistent and pervasive issue in many parts of the world, particularly in developing countries, has a significant impact on the political, social, and economic spheres. Because financial corruption interacts with the population, the Lotka-Volterra model allows for the examination of how degrees of financial corruption and population segments influence each other over time. The model can be used to identify potential tipping points when the level of financial corruption may lead to significant societal consequences, such as widespread unrest, economic downturns, or changes in population behavior. Understanding these dynamics is essential for formulating strategies to counteract financial corruption and promote stability and prosperity.

The analysis of differential equations and their solutions is essential for understanding numerous phenomena across science, engineering, and mathematics. A critical part of this analysis involves verifying the existence, uniqueness, and stability of solutions under specific conditions. An important criterion that significantly contributes to these investigations is the Lipschitz condition, which provides a measure of how quickly a function can vary with its variables.

In this paper, we introduces a nonlinear Lotka-Volterra model depicts the relationship between population and the level of corruption. The main tools used in the study are the Picard approximation iteration and the principles of Ulam stability. The proposed model is governed by the following system of differential equations:

$$\frac{d\Phi(\tau)}{d\tau} = \xi_1\Phi(\tau) - \xi_2\Phi(\tau)\Psi(\tau) - \alpha\Phi^2(\tau),$$
$$\frac{d\Psi(\tau)}{d\tau} = -\rho_1\Psi(\tau) + \rho_2\Phi(\tau)\Psi(\tau) - \beta \ln\big(\Psi(\tau)\big). \qquad (1.1)$$
$$\Phi_0(\tau) = \Phi_0, \Psi_0(\tau) = \Psi_0.$$

Here, $\Phi(\tau)$ represents a time-dependent social or demographic indicator, while $\Psi(\tau)$ denotes the level of corruption. The constants $\xi_1, \xi_2, \alpha, \rho_1, \rho_2$, and β are fixed parameters characterizing the internal dynamics and interaction coefficients of the system. Specifically, the term $-\alpha\Phi^2(\tau)$ models intrinsic limitations within the population, such as resource constraints or saturation effects, whereas the logarithmic term $-\beta\, ln\,(\Psi(\tau))$ reflects the diminishing impact of increasing corruption on the system.

Due to the presence of the logarithmic function, the domain of the model requires the condition $\Psi(\tau) > 0$ to ensure the regularity and well-definedness of the system. Assuming strictly positive initial conditions and bounded solutions, the local existence and uniqueness of solutions are established through the method of Picard successive approximations. The continuity and local Lipschitz continuity of the right-hand side functions satisfy the criteria of the Picard–Lindelöf theorem, thereby guaranteeing the uniqueness of the solution in a local time interval.

In addition, the model's sensitivity to small perturbations in initial conditions is examined within the frameworks of Ulam, Ulam–Hyers, and Ulam–Hyers–Rassias stability. The results demonstrate that the system exhibits robustness with respect to such perturbations and that the solution is stable in the Ulam sense. These findings underscore the reliability of the model under realistic assumptions and contribute to the theoretical foundation for future applications in socio-economic policy modeling.

2. Model Description

System (1.1) models the dynamic interaction between honest individuals in society, denoted by $\Phi(\tau)$, and the level of corruption, denoted by $\Psi(\tau)$. The system of differential equations represents how these two populations evolve over time under various influences.

$\Phi(\tau)$ represents the number or proportion of honest individuals in society at time τ, while $\Psi(\tau)$ represents the level of corruption in society at time τ. ξ_1 is the natural growth rate of honest individuals, ξ_2 is the rate at which corruption influences and reduces honesty, α is a saturation effect that limits the growth of honesty, possibly due to social or economic constraints, ρ_1 is the natural decline rate of corruption, ρ_2 is the reinforcement effect where corruption increases due to interaction with honest individuals, and β is a nonlinear term modeling the logarithmic impact of corruption reduction mechanisms.

Equation 1: Honest Individuals Dynamics

$$\frac{d\Phi(\tau)}{d\tau} = \xi_1\Phi(\tau) - \xi_2\Phi(\tau)\Psi(\tau) - \alpha\Phi^2(\tau).$$

The term $\xi_1\Phi(\tau)$ represents the natural growth of honesty. The term $-\xi_2\Phi(\tau)\Psi(\tau)$ represents how corruption negatively impacts honesty, possibly through social pressure or institutional decay. The term $-\alpha\Phi^2(\tau)$ accounts for a saturation effect where honesty cannot grow indefinitely due to societal limitations.

Equation 2: Corruption Dynamics

$$\frac{d\Psi(\tau)}{d\tau} = -\rho_1\Psi(\tau) + \rho_2\Phi(\tau)\Psi(\tau) - \beta\ln\big(\Psi(\tau)\big).$$

The term $-\rho_1\Psi(\tau)$ represents the natural decay of corruption, such as due to law enforcement or societal resistance. The term $\rho_2\Phi(\tau)\Psi(\tau)$ represents the reinforcement of corruption through interaction with honest individuals (e.g., bribery converting honest individuals into corrupt ones). The term $-\beta\ln\big(\Psi(\tau)\big)$ models the nonlinear effect of anti-corruption measures, where corruption reduces, but at a decreasing rate.

The model suggests conditions under which corruption can be eliminated ($\Psi\to 0$) or persists at a steady level. Depending on the parameters, a small increase in corruption influence ξ_2, ρ_2 can lead to a sudden rise in corruption, while stronger anti-corruption measures β can suppress it. Increasing ρ_1 or decreasing ρ_2 can shift society towards a lower-corruption state.

3. Preliminaries

***Definition* 3.1** [9]. Let $f(x,y)$ be a function defined on the set $(a,b)XG$, where $G\subset R$. The function $f(x,y)$ is said to satisfy the Lipschitz condition with respect to the second variable if, for all $x\in(a,b)$ and for any $y_1, y_2\in G$, the following inequality holds:

$$|f(x,\, y_1) - f(x,\, y_2)| \le \xi|y_1 - y_2|.$$

***Definition* 3.2** [2]. Let $\{f_m(t)\}_{m=0}^{\infty}$ be a sequence of functions defined on a set $E \subseteq R^1$. We say that $\{f_m(t)\}_{m=0}^{\infty}$ converges uniformly to the limit function $f(t)$ on E if for every $\mathcal{E} > 0$, there exists a positive integer N such that for all $m \ge N$ and for all $t\in E$, the following condition is satisfied:

$$|f_m(t) - f(t)| < \mathcal{E}.$$

***Definition* 3.3** [15]. A differential equation is said to be Ulam-Hyers-Rassias stable with respect to φ if there exists a positive constanit $c_{f,\varphi} > 0$ such that for every $\mathcal{E} > 0$ and for every solution $y\in C^1([a,b),\mathbb{B})$, there exists a solution $x\in C^1([a,b),\mathbb{B})$ of the given equation satisfying

$$|y(t) - x(t)| \le c_{f,\varphi}\varepsilon\varphi(t), \qquad \forall t\in[a,b).$$

***Definition* 3.4** [6]. Let E_1 be a group and E_2 be a quasi-normed space. If the functions $F, G : E_1 \longrightarrow E_2$ satisfy the inequality

$$d[F(x+y+z) + F(x-y) + F(y-z) + F(x-z), G(x) + G(y) + G(z)] \le h(x,y,z),$$

where h is constant, it is referred to as Hyers-Ulam Stability. Rassias introduced an inequality of the following from:

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p),$$

where $\theta \ge 0$, which is know as Hyers-Ulam-Rassias stability (a generalized form of Hyers-Ulam stability). If the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon,$$

is replaced by

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p),$$

where $\varepsilon > 0$, it is referred to as generalized Hyers-Ulam-Rassias stability.

4. Main Results

Lemma 4.1: If the functions $\Phi(\tau)$ and $\Psi(\tau)$ are bounded and $\Phi(\tau)$ is bounded away from zero, then the functions $\mathcal{F}(\tau,\Phi,\Psi)$ and $g(\tau,\Phi,\Psi)$ satisfies lipschitz condition where:

$$\mathcal{F}(\tau,\Phi,\Psi) = \xi_1\Phi(\tau) - \xi_2\Phi(\tau)\Psi(\tau) - \alpha\Phi^2(\tau),$$

$$g(\tau,\Phi,\Psi) = -\rho_1\Psi(\tau) + \rho_2\Phi(\tau)\Psi(\tau) - \beta l\,n\big(\Psi(\tau)\big).$$

Proof: A function is satisfies lipschitz condition if there exists a constant $(\mathcal{L})$ such that for all (τ) in the domain:

$$\|\mathcal{F}(\tau,\Phi_1,\Psi_1) - \mathcal{F}(\tau,\Phi_2,\Psi_2)\| \le \mathcal{L}_1(\|\Phi_1-\Phi_2\| + \|\Psi_1-\Psi_2\|).$$

Similarly, the same condition must hold for g.

The partial derivative of $\mathcal{F}$ with respect to (Φ) is:

$$\frac{d\mathcal{F}}{d\Phi} = \xi_1 - \xi_2\Psi(\tau) - 2\,\alpha\,\Phi(\tau),$$

with respect to (Ψ) is:

$$\frac{d\mathcal{F}}{d\Psi} = -\xi_2\Phi(\tau).$$

These derivative are linear in Φ and Ψ. If $\Phi(\tau)$ and $\Psi(\tau)$ are bounded, say $|\Phi(\tau)| \le \mathcal{M}_\Phi$ and $|\Psi(\tau)| \le \mathcal{M}_\Psi$ then:

$$\left|\frac{d\mathcal{F}}{d\Phi}\right| \le |\xi_1| - |\xi_2|\mathcal{M}_\Psi - 2|\,\alpha|\,\mathcal{M}_\Phi,$$

$$\left|\frac{d\mathcal{F}}{d\Psi}\right| \le |\xi_2|\mathcal{M}_\Phi.$$

Since both partial derivatives are bounded under assumption that (Φ and Ψ) are bounded, $(\mathcal{F})$ satisfies lipschitz condition on any compact domain.

The partial derivatives of g with respect to (Φ) is:

$$\frac{dg}{d\Phi} = \rho_2\Psi(\tau),$$

with respect to (Ψ) is:

$$\frac{dg}{d\Psi} = -\rho_1 + \rho_2\Phi(\tau) - \frac{\beta}{\Psi(\tau)}\,.$$

The term $\frac{dg}{d\Phi}$ is linear in Ψ. And its bound is:

$$\left|\frac{dg}{d\Phi}\right| \le |\rho_2|\mathcal{M}_\Psi.$$

However, the term$(\frac{dg}{d\Psi})$ includes$(-\frac{\beta}{\Psi(\tau)})$, which becomes unbounded as $(\Psi(\tau)\to 0)$. To ensure lipschitz continuity, $\Psi(\tau)$ must be bounded away from zero, there exists a positive constant $\mathcal{M}_\Psi$ such that$(\Psi(\tau) \ge \mathcal{M}_\Psi > 0)$, under this condition:

$$\left|\frac{dg}{d\Psi}\right| \le |\rho_1| + |\rho_2|\mathcal{M}_\Phi + \frac{|\beta|}{\mathcal{M}_\Psi}.$$

Since all derivatives are bounded under the given assumptions, ($\mathcal{g}$) satisfies lipschitz condition. If ($\Phi(\tau)$ and $\Psi(\tau)$ are bounded and $\Psi(\tau)$ is bounded away from zero, then $\mathcal{F}(\tau,\Phi,\Psi)$ and $\mathcal{g}(\tau,\Phi,\Psi)$ satisfy the lipschitz condition. ■

Theorem 4.1: Let $\mathcal{F}: \mathbb{I}\times\mathbb{J}\times\mathbb{K}\longrightarrow\mathbb{R}$ and $\mathcal{g}: \mathbb{I}\times\mathbb{J}\times\mathbb{K}\longrightarrow\mathbb{R}$ are two continuous functions, where $\mathbb{I}\times\mathbb{J}\times\mathbb{K}$ is an open domain in $\mathbb{R}^3$. If there exist constants $\mathcal{L}_1,\mathcal{L}_2>0$ such that :

i) $|\mathcal{F}(\tau,\Phi_1,\Psi_1)-\mathcal{F}(\tau,\Phi_2,\Psi_2)|\le\mathcal{L}_1(|\Phi_1-\Phi_2|+|\Psi_1-\Psi_2|)$,

ii) $|\mathcal{g}(\tau,\Phi_1,\Psi_1)-\mathcal{g}(\tau,\Phi_2,\Psi_2)|\le\mathcal{L}_2(|\Phi_1-\Phi_2|+|\Psi_1-\Psi_2|)$,

for $\forall\tau\in\mathbb{I}$, $\Phi_1,\Phi_2\in\mathbb{J}$, $\Psi_1,\Psi_2\in\mathbb{K}$.

Then for each interior point (τ_0,Φ_0,Ψ_0) in $\mathbb{I}\times\mathbb{J}\times\mathbb{K}$ there exist an interval $\mathbb{I}_{\hbar}=(\tau_0-\hbar,\tau_0+\hbar)$ and there exist a unique solution for system (1.1) on $\mathbb{I}_{\hbar}$.

Proof:

(a) Find the interval $\mathbb{I}_{\hbar}=[\tau_0,-\hbar,\tau_0,+\hbar]$

Since $(\tau_0,\Phi_0,\Psi_0)\in\mathbb{I}\times\mathbb{J}\times\mathbb{K}$, then there is a closed neighborhood $\mathbb{I}_{\mathfrak{a}}\times\mathbb{J}_{\mathfrak{b}}\times\mathbb{K}_{\mathfrak{c}}=[\tau_0-\mathfrak{a},\tau_0+\mathfrak{a}]\times[\Phi_0-\mathfrak{b},\Phi_0+\mathfrak{b}]\times[\Psi_0-\mathfrak{c},\Psi_0+\mathfrak{c}]$ such that $\mathcal{F}$ and $\mathcal{g}$ are continuous on it, and there exists $\mathcal{M},\mathcal{N}>0$ such that:

$$\begin{cases}|\mathcal{F}(\tau,\Phi,\Psi)|\le\mathcal{M}\,;\\|\mathcal{g}(\tau,\Phi,\Psi)|\le\mathcal{N}\,;\end{cases}\qquad\forall(\tau,\Phi,\Psi)\in\mathbb{I}_{\mathfrak{a}}\times\mathbb{J}_{\mathfrak{b}}\times\mathbb{K}_{\mathfrak{c}}.$$

We choose

$$\hbar=\min\left(\mathfrak{a},\frac{\mathfrak{b}}{\mathcal{M}},\frac{\mathfrak{c}}{\mathcal{N}}\right).$$

Let the sequences Φ_n and Ψ_n be defined on the interval $\mathbb{I}_{\hbar}$ as follows:

$$\Phi_n(\tau)=\Phi_0+\int_{\tau_0}^{\tau}\mathcal{F}(s,\Phi_{n-1}(s),\Psi_{n-1}(s))ds,\qquad\forall n\ge1,$$

$$\Psi_n(\tau)=\Psi_0+\int_{\tau_0}^{\tau}\mathcal{g}(s,\Phi_{n-1}(s),\Psi_{n-1}(s))ds,\qquad\forall n\ge1,$$

with the initial conditions:

$$\Phi_0(\tau)=\Phi_0,\qquad\Psi_0(\tau)=\Psi_0.$$

For each $\tau\in\mathbb{I}_{\hbar}$, we have $\Phi_n(\tau)\in\mathbb{J}_{\mathfrak{b}}$, $\Psi_n(\tau)\in\mathbb{K}_{\mathfrak{c}}$ $\forall n\ge1$; that is $(\tau,\Phi_n(\tau),\Psi_n(\tau))\in\mathbb{I}_{\hbar}\times\mathbb{J}_{\mathfrak{b}}\times\mathbb{K}_{\mathfrak{c}}$ $\forall n\ge1$, for n =1:

$$\Phi_1(\tau)=\Phi_0+\int_{\tau_0}^{\tau}\mathcal{F}(s,\Phi(s),\Psi(s))ds,$$

Applying the bound on $\mathcal{F}$:

$$|\Phi_1(\tau)-\Phi_0|=\left|\int_{\tau_0}^{\tau}\mathcal{F}(s,\Phi_0(s),\Psi_0(s))ds\right|$$

$$|\Phi_1(\tau)-\Phi_0|\le\int_{\tau_0}^{\tau}\left|\mathcal{F}\big(s,\Phi_0(s),\Psi_0(s)\big)\right|ds$$

$$\le\mathcal{M}\int_{\tau_0}^{\tau}ds\le\mathcal{M}|\tau-\tau_0|\le\mathcal{M}\hbar\le\mathfrak{b}.$$

Similarly, for $\Psi_1(\tau)$:

$$\Psi_1(\tau)=\Psi_0+\int_{\tau_0}^{\tau}\mathcal{g}(s,\Phi(s),\Psi(s))ds,$$

$$|\Psi_1(\tau)-\Psi_0|=\left|\int_{\tau_0}^{\tau}\mathcal{g}\big(s,\Phi_0(s),\Psi_0(s)\big)ds\right|$$

$$|\Psi_1(\tau)-\Psi_0|\le\int_{\tau_0}^{\tau}\left|\mathcal{g}\big(s,\Phi_0(s),\Psi_0(s)\big)\right|ds$$

$$\le\mathcal{N}\int_{\tau_0}^{\tau}ds\le\mathcal{N}|\tau-\tau_0|\le\mathcal{N}\hbar\le\mathfrak{c}.$$

Thus, for every $\tau\in\mathbb{I}_{\hbar}$, we have $\Phi_1(\tau)\in\mathbb{J}_{\mathfrak{b}}$ *and* $\Psi_1(\tau)\in\mathbb{K}_{\mathfrak{c}}$.

Assume that the relationship is valid for n , i.e:

$$|\Phi_n(\tau)-\Phi_0|\le\mathfrak{b}.$$
$$|\Psi_n(\tau)-\Psi_0|\le\mathfrak{c}.$$

We prove the validity of the relationship at (n +1). We have for each $\tau\in\mathbb{I}_{\hbar}$

$$\Phi_{n+1}(\tau)=\Phi_0+\int_{\tau_0}^{\tau}\mathcal{F}(s,\Phi_n(s),\Psi_n(s))ds,$$

$$|\Phi_{n+1}(\tau)-\Phi_0|=\left|\int_{\tau_0}^{\tau}\mathcal{F}(s,\Phi_n(s),\Psi_n(s))ds\right|$$

$$|\Phi_{n+1}(\tau)-\Phi_0|\le\int_{\tau_0}^{\tau}\left|\mathcal{F}\big(s,\Phi_n(s),\Psi_n(s)\big)\right|ds$$

$$\le\mathcal{M}\int_{\tau_0}^{\tau}ds\le\mathcal{M}|\tau-\tau_0|\le\mathcal{M}\hbar\le\mathfrak{b}.$$

Similarly:

$$\Psi_{n+1}(\tau)=\Psi_0+\int_{\tau_0}^{\tau}\mathcal{g}\big(s,\Phi_n(s),\Psi_n(s)\big)ds,$$

$$|\Psi_{n+1}(\tau)-\Psi_0|=\left|\int_{\tau_0}^{\tau}\mathcal{g}(s,\Phi_n(s),\Psi_n(s))ds\right|$$

$$|\Psi_{n+1}(\tau)-\Psi_0|\le\int_{\tau_0}^{\tau}\left|\mathcal{g}\big(s,\Phi_n(s),\Psi_n(s)\big)\right|ds$$

$$\le\mathcal{N}\int_{\tau_0}^{\tau}ds\le\mathcal{N}|\tau-\tau_0|\le\mathcal{N}\hbar\le\mathfrak{c}.$$

Hence: $\Phi_{n+1}(\tau)\in\mathbb{J}_{\mathfrak{b}}$ *and* $\Psi_{n+1}(\tau)\in\mathbb{K}_{\mathfrak{c}}$ therefore:

$$\Phi_n(\tau)\in\mathbb{J}_{\mathfrak{b}}\qquad\forall n=1,2,\ldots.$$
$$\Psi_n(\tau)\in\mathbb{K}_{\mathfrak{c}}\qquad\forall n=1,2,\ldots.$$

(b) The sequences $(\Phi_n(\tau),\Psi_n(\tau))$ convergence over $\mathbb{I}_{\hbar}$ $\forall n\ge1$.

It is sufficient to show that for each $\tau\in\mathbb{I}_{\hbar}$ the series $\sum_n(\Phi_{n+1}(\tau)-\Phi_n(\tau))$and $\sum_n(\Psi_{n+1}(\tau)-\Psi_n(\tau))$ convergences.

For $|\tau-\tau_0|\le\hbar$ we have:

$$\Phi_1(\tau)=\Phi_0(\tau)+\int_{\tau_0}^{\tau}\mathcal{F}(s,\Phi(s),\Psi(s))ds,$$

$$|\Phi_1(\tau)-\Phi_0(\tau)|=\left|\int_{\tau_0}^{\tau}\mathcal{F}(s,\Phi_0(s),\Psi_0(s))ds\right|$$

$$|\Phi_1(\tau)-\Phi_0(\tau)|\le\int_{\tau_0}^{\tau}\left|\mathcal{F}\big(s,\Phi_0(s),\Psi_0(s)\big)\right|ds$$

$$\le\mathcal{M}\int_{\tau_0}^{\tau}ds\le\mathcal{M}|\tau-\tau_0|\le\mathcal{M}\hbar.$$

Similarly:

$$|\Psi_1(\tau)-\Psi_0(\tau)| \leq \mathcal{N}\hbar.$$

$$|\Phi_2(\tau)-\Phi_1(\tau)| = \left|\Phi_0 + \int_{\tau_0}^{\tau} \mathcal{F}(s,\Phi_1(s),\Psi_1(s))ds - \Phi_0 - \int_{\tau_0}^{\tau} \mathcal{F}(s,\Phi_0(s),\Psi_0(s))ds\right|$$

$$\leq \int_{\tau_0}^{\tau} |\mathcal{F}(s,\Phi_1(s),\Psi_1(s)) - \mathcal{F}(s,\Phi_0(s),\Psi_0(s))|ds$$

$$\leq \mathcal{L}_1 \int_{\tau_0}^{\tau} (|\Phi_1(s)-\Phi_0(s)| + |\Psi_1(s)-\Psi_0(s)|)ds$$

$$\leq \mathcal{L}_1 \int_{\tau_0}^{\tau} (\mathcal{M}|s-\tau_0| + \mathcal{N}|s-\tau_0|)ds$$

$$\leq \mathcal{L}_1(\mathcal{M}+\mathcal{N}) \int_{\tau_0}^{\tau} |s-\tau_0|ds$$

$$\leq \mathcal{L}_1(\mathcal{M}+\mathcal{N})\left[\frac{(s-\tau_0)^2}{2}\right]_{\tau_0}^{\tau}$$

$$= \frac{\mathcal{L}_1(\mathcal{M}+\mathcal{N})\hbar^2}{2}.$$

Similarly:

$$|\Psi_2(\tau)-\Psi_1(\tau)| = \frac{\mathcal{L}_2(\mathcal{M}+\mathcal{N})\hbar^2}{2}.$$

Thus, we obtain:

$$|\Phi_{n+1}(\tau)-\Phi_n(\tau)| \leq \frac{\mathcal{L}_1(\mathcal{L}_1+\mathcal{L}_2)^{n-1}(\mathcal{M}+\mathcal{N})\hbar^{n+1}}{(n+1)!}.$$

$$|\Psi_{n+1}(\tau)-\Psi_n(\tau)| \leq \frac{\mathcal{L}_2(\mathcal{L}_1+\mathcal{L}_2)^{n-1}(\mathcal{M}+\mathcal{N})\hbar^{n+1}}{(n+1)!}.$$

Let $\mathcal{V}(\tau) = \begin{pmatrix} |\Phi_{n+1}-\Phi_n| \\ |\Psi_{n+1}-\Psi_n| \end{pmatrix}$,

$$\delta(\tau) = \begin{pmatrix} \frac{\mathcal{L}_1(\mathcal{L}_1+\mathcal{L}_2)^{n-1}(\mathcal{M}+\mathcal{N})\hbar^{n+1}}{(n+1)!} \\ \frac{\mathcal{L}_2(\mathcal{L}_1+\mathcal{L}_2)^{n-1}(\mathcal{M}+\mathcal{N})\hbar^{n+1}}{(n+1)!} \end{pmatrix}, \quad \forall n \geq 0.$$

$\Rightarrow \quad \mathcal{V}(\tau) \leq \delta(\tau).$

The series $\delta(\tau)$ is convergent, since:

$$u_n = \frac{\mathcal{L}_1(\mathcal{L}_1+\mathcal{L}_2)^{n-1}(\mathcal{M}+\mathcal{N})\hbar^{n+1}}{(n+1)!}.$$

and

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{\mathcal{L}_1(\mathcal{L}_1+\mathcal{L}_2)^{n}(\mathcal{M}+\mathcal{N})\hbar^{n+2}/(n+2)!}{\mathcal{L}_1(\mathcal{L}_1+\mathcal{L}_2)^{n-1}(\mathcal{M}+\mathcal{N})\hbar^{n+1}/(n+1)!}\right)$$

$$= \lim_{n\to\infty} \frac{(\mathcal{L}_1+\mathcal{L}_2)\hbar}{(n+2)} = 0 < 1.$$

Since $\delta(\tau)$ is converges, $\mathcal{V}(\tau)$ also converges, meaning that $\Phi_n(\tau)$ and $\Psi_n(\tau)$ are convergent.

Let $\Phi(\tau)$ and $\Psi(\tau)$ be their limits $\Phi_n(\tau)$ and $\Psi_n(\tau)$ respectively, that is for each $\tau \in \mathbb{I}_\hbar$.

$$\lim_{n\to\infty} |\Phi_n(\tau)-\Phi(\tau)| = 0$$

$\Rightarrow \forall \tau \in \mathbb{I}_\hbar$ and $\xi_1, \xi_2 > 0$, there exists $n_0 \in \mathcal{N}$ where $n \geq n_0$.

$$|\Phi_n(\tau)-\Phi(\tau)| < \frac{\xi_1}{\mathcal{M}\hbar}, \qquad |\Psi_n(\tau)-\Psi(\tau)| < \frac{\xi_2}{\mathcal{N}\hbar}.$$

And hence $\tau \in \mathbb{I}_\hbar$

$$\left|\int_{\tau_0}^{\tau} \mathcal{F}(s,\Phi_n(s),\Psi_n(s))ds - \int_{\tau_0}^{\tau} \mathcal{F}(s,\Phi(s),\Psi(s))ds\right|$$

$$\leq \mathcal{L} \int_{t_0}^{t} (|\Phi_n(s)-\Phi(s)| + |\Psi_n(s)-\Psi(s)|)ds$$

$$\leq \mathcal{L} \int_{\tau_0}^{\tau} \left(\frac{\xi_1}{\mathcal{M}\hbar} + \frac{\xi_2}{\mathcal{N}\hbar}\right)ds$$

$$\leq \mathcal{L}\left(\frac{\xi_1}{\mathcal{M}\hbar} + \frac{\xi_2}{\mathcal{N}\hbar}\right)\hbar \leq \xi_1 + \xi_2.$$

$$\Rightarrow \lim_{n\to\infty} \int_{\tau_0}^{\tau} \mathcal{F}(s,\Phi_n(s),\Psi_n(s))ds = \int_{\tau_0}^{\tau} \mathcal{F}(s,\Phi(s),\Psi(s))ds$$

Taking the limit of the equation:

$$\Phi_n(\tau) = \Phi_0 + \int_{\tau_0}^{\tau} \mathcal{F}(s,\Phi_{n-1}(s),\Psi_{n-1}(s))ds,$$

$$\Phi(\tau) = \Phi_0 + \int_{\tau_0}^{\tau} \mathcal{F}(s,\Phi(s),\Psi(s))ds,$$

$$\Psi(\tau) = \Psi_0 + \int_{\tau_0}^{\tau} g(s,\Phi(s),\Psi(s))ds,$$

And we can calculate from part (a):

$$(\tau,\Phi(\tau),\Psi(\tau)) \in \mathbb{I}_\hbar \times \mathbb{J}_b \times \mathbb{K}_c.$$

It remains to prove that the function (Φ,Ψ) are continuous on the interval $\mathbb{I}_\hbar$.

For any $\xi_1, \xi_2 > 0$, let $\delta = \frac{\mathcal{E}_1}{\mathcal{M}}$ and $\delta = \frac{\mathcal{E}_2}{\mathcal{N}}$ and for every $\tau_1, \tau_2 \in \mathbb{I}_\hbar$ where $|\tau_1 - \tau_2| < \delta$, we have:

$$|\Phi(\tau_1)-\Phi(\tau_2)| = \left|\int_{\tau_0}^{\tau_1} \mathcal{F}(s,\Phi(s),\Psi(s))ds - \int_{\tau_0}^{\tau_2} \mathcal{F}(s,\Phi(s),\Psi(s))ds\right|$$

$$\leq \left|\int_{\tau_1}^{\tau_2} \mathcal{F}(s,\Phi(s),\Psi(s))ds\right| \leq \int_{\tau_1}^{\tau_2} |\mathcal{F}(s,\Phi(s),\Psi(s))|\, ds$$

$$\leq \mathcal{M}\int_{\tau_1}^{\tau_2} ds \leq \mathcal{M}|\tau_2 - \tau_1| \leq \mathcal{M}\delta = \mathcal{M}\frac{\mathcal{E}_1}{\mathcal{M}} = \mathcal{E}_1.$$

$$|\Psi(\tau_1)-\Psi(\tau_2)| = \left|\int_{\tau_0}^{\tau_1} g(s,\Phi(s),\Psi(s))ds - \int_{\tau_0}^{\tau_2} g(s,\Phi(s),\Psi(s))ds\right|$$

$$\leq \left|\int_{\tau_1}^{\tau_2} g(s,\Phi(s),\Psi(s))ds\right| \leq \int_{\tau_1}^{\tau_2} |g(s,\Phi(s),\Psi(s))|\, ds$$

$$\leq \mathcal{N}\int_{\tau_1}^{\tau_2} ds \leq \mathcal{N}|\tau_2 - \tau_1| \leq \mathcal{N}\delta = \mathcal{N}\frac{\mathcal{E}_2}{\mathcal{N}} = \mathcal{E}_2.$$

Therefore (Φ,Ψ) are uniformly continuous on $\mathbb{I}_\hbar$ and are connected over the same period $\mathbb{I}_\hbar$.

(c): We prove that the solution (Φ,Ψ) is unique. Assume that $(\widetilde{\Phi}(\tau),\widetilde{\Psi}(\tau))$ is another differentiable function defined on

$[\tau_0, \tau_0 + \hbar]$ such that :

$$\frac{d\widetilde{\Phi}(\tau)}{d\tau} = \mathcal{F}\left(\tau, \widetilde{\Phi}(\tau), \widetilde{\Psi}(\tau)\right),$$
$$\frac{d\widetilde{\Psi}(\tau)}{d\tau} = g\left(\tau, \widetilde{\Phi}(\tau), \widetilde{\Psi}(\tau)\right),$$

With the initial conditions:

$$\widetilde{\Phi}(\tau_0) = \Phi_0, \qquad \widetilde{\Psi}(\tau_0) = \Psi_0.$$

Then, certainly:

$$\left|\widetilde{\Phi}(\tau) - \Phi_0\right| < \mathfrak{b}, \qquad \left|\widetilde{\Psi}(\tau) - \Psi_0\right| < \mathfrak{c}.$$

On some interval $[\tau_0, \tau_0 + \delta]$, let τ_1 be such that:

$\left|\widetilde{\Phi}(\tau) - \Phi_0\right| < \mathfrak{b}$ and $\left|\widetilde{\Psi}(\tau) - \Psi_0\right| < \mathfrak{c}$ for $\tau_0 < \tau < \tau_1$ and $\left|\widetilde{\Phi}(\tau) - \Phi_0\right| = \mathfrak{b}$ and $\left|\widetilde{\Psi}(\tau) - \Psi_0\right| = \mathfrak{c}$. Suppose that $\tau_1 < \tau_0 + \hbar$. Then, $\hbar = \min(\mathfrak{a}, \frac{\mathfrak{b}}{\mathcal{M}}, \frac{\mathfrak{c}}{\mathcal{N}})$.

$$\mathcal{M}_1 = \left|\frac{\widetilde{\Phi}(\tau_1) - \Phi_0}{\tau_1 - \tau_0}\right| = \frac{\mathfrak{b}}{\tau_1 - \tau_0} > \frac{\mathfrak{b}}{\hbar} \geq \mathcal{M},$$
$$\mathcal{N}_2 = \left|\frac{\widetilde{\Psi}(\tau_1) - \Psi_0}{\tau_1 - \tau_0}\right| = \frac{\mathfrak{c}}{\tau_1 - \tau_0} > \frac{\mathfrak{c}}{\hbar} \geq \mathcal{N}.$$

By the mean-value theorem, there exist $(\varepsilon_1, \varepsilon_2)$ with $\tau_0 < \varepsilon_1, \varepsilon_2 < \tau_1$, such that:

$$\mathcal{M}_1 = \left|\widetilde{\Phi}(\varepsilon_1)\right| = \left|\mathcal{F}(\varepsilon_1, \widetilde{\Phi}(\varepsilon_1), \widetilde{\Psi}(\varepsilon_1))\right| \leq \mathcal{M},$$
$$\mathcal{N}_1 = \left|\widetilde{\Psi}(\varepsilon_2)\right| = \left|g(\varepsilon_2, \widetilde{\Phi}(\varepsilon_2), \widetilde{\Psi}(\varepsilon_2))\right| \leq \mathcal{N}.$$

This is a contradiction. Thus $\tau_1 \geq \tau_0 + \hbar$ and the inequalities hold for $\tau_0 \leq \tau \leq \tau_0 + \hbar$, so:

$$\left|\widetilde{\Phi}(\tau) - \Phi_0\right| \leq \mathfrak{b}, \qquad \left|\widetilde{\Psi}(\tau) - \Psi_0\right| \leq \mathfrak{c}.$$

on the interval $\tau_0 \leq \tau \leq \tau_0 + \hbar$.

Since $(\widetilde{\Phi}, \widetilde{\Psi})$ is a solution of system (1.1) on $(\tau_0, \tau_0 + \hbar)$ such that $(\widetilde{\Phi}(\tau_0) - \Phi_0)$ and $(\widetilde{\Psi}(\tau_0) - \Psi_0)$, we see that $(\widetilde{\Phi}, \widetilde{\Psi})$ satisfies the integral equation:

$$\widetilde{\Phi}(\tau) = \Phi_0 + \int_{\tau_0}^{\tau} \mathcal{F}\left(s, \widetilde{\Phi}(s), \widetilde{\Psi}(s)\right) ds,$$
$$\widetilde{\Psi}(\tau) = \Psi_0 + \int_{\tau_0}^{\tau} g\left(s, \widetilde{\Phi}(s), \widetilde{\Psi}(s)\right) ds.$$

on $[\tau_0, \tau_0 + \hbar]$. We shall now prove by mathematical induction that :

$$\left|\widetilde{\Phi}(\tau) - \Phi_n(\tau)\right| \leq \frac{\mathcal{L}_1(\mathcal{L}_1 + \mathcal{L}_2)^{n-1}(\mathfrak{b} + \mathfrak{c})(\tau - \tau_0)^n}{n!} \leq \frac{\mathcal{L}_1(\mathcal{L}_1 + \mathcal{L}_2)^{n-1}(\mathfrak{b} + \mathfrak{c})\hbar^n}{n!},$$
$$\left|\widetilde{\Psi}(\tau) - \Psi_n(\tau)\right| \leq \frac{\mathcal{L}_2(\mathcal{L}_1 + \mathcal{L}_2)^{n-1}(\mathfrak{b} + \mathfrak{c})(\tau - \tau_0)^n}{n!} \leq \frac{\mathcal{L}_2(\mathcal{L}_1 + \mathcal{L}_2)^{n-1}(\mathfrak{b} + \mathfrak{c})\hbar^n}{n!}.$$

on $[\tau_0, \tau_0 + \hbar]$. We thus assume that:

$$\left|\widetilde{\Phi}(\tau) - \Phi_{n-1}(\tau)\right| \leq \frac{\mathcal{L}_1(\mathcal{L}_1 + \mathcal{L}_2)^{n-2}(\mathfrak{b} + \mathfrak{c})(\tau - \tau_0)^{n-1}}{(n-1)!} \leq \frac{\mathcal{L}_1(\mathcal{L}_1 + \mathcal{L}_2)^{n-2}(\mathfrak{b} + \mathfrak{c})\,\hbar^{n-1}}{(n-1)!}.$$
$$\left|\widetilde{\Psi}(\tau) - \Psi_{n-1}(\tau)\right| \leq \frac{\mathcal{L}_2(\mathcal{L}_1 + \mathcal{L}_2)^{n-2}(\mathfrak{b} + \mathfrak{c})(\tau - \tau_0)^{n-1}}{(n-1)!} \leq \frac{\mathcal{L}_2(\mathcal{L}_1 + \mathcal{L}_2)^{n-2}(\mathfrak{b} + \mathfrak{c})\hbar^{n-1}}{(n-1)!}.$$

on $[\tau_0, \tau_0 + \hbar]$. We have:

$$\left|\widetilde{\Phi}(\tau) - \Phi_1(\tau)\right| \leq \int_{\tau_0}^{\tau}\left|\mathcal{F}(s, \widetilde{\Phi}(s), \widetilde{\Psi}(s)) - \mathcal{F}(s, \Phi_0, \Psi_0)\right| ds$$
$$\leq \mathcal{L}_1 \int_{\tau_0}^{\tau} \left(\left|\widetilde{\Phi}(s) - \Phi_0\right| + \left|\widetilde{\Psi}(s) - \Psi_0\right|\right) ds$$
$$\leq \mathcal{L}_1(\mathfrak{b} + \mathfrak{c})(\tau - \tau_0) \leq \mathcal{L}_1(\mathfrak{b} + \mathfrak{c})\hbar.$$

$$\left|\widetilde{\Phi}(\tau) - \Phi_2(\tau)\right| \leq \int_{\tau_0}^{\tau}\left|\mathcal{F}(s, \widetilde{\Phi}(s), \widetilde{\Psi}(s)) - \mathcal{F}(s, \Phi_1, \Psi_1)\right| ds$$
$$\leq \mathcal{L}_1 \int_{\tau_0}^{\tau} \left(\left|\widetilde{\Phi}(s) - \Phi_1\right| + \left|\widetilde{\Psi}(s) - \Psi_1\right|\right) ds$$
$$\leq \mathcal{L}_1 \int_{\tau_0}^{\tau} \left(\mathcal{L}_1(\mathfrak{b} + \mathfrak{c})\hbar + \mathcal{L}_2(\mathfrak{b} + \mathfrak{c})\hbar\right) ds$$
$$\leq \frac{\mathcal{L}_1^{\,2}(\mathfrak{b} + \mathfrak{c})\hbar^2}{2} + \frac{\mathcal{L}_1\mathcal{L}_2(\mathfrak{b} + \mathfrak{c})\hbar^2}{2}$$
$$\leq \frac{\mathcal{L}_1(\mathcal{L}_1 + \mathcal{L}_2)(\mathfrak{b} + \mathfrak{c})\hbar^2}{2}.$$

Similarly:

$$\left|\widetilde{\Psi}(\tau) - \Psi_1(\tau)\right| \leq \int_{\tau_0}^{\tau}\left|g(s, \widetilde{\Phi}(s), \widetilde{\Psi}(s)) - g(s, \Phi_0, \Psi_0)\right| ds$$
$$\leq \mathcal{L}_2 \int_{\tau_0}^{\tau} \left(\left|\widetilde{\Phi}(s) - \Phi_0\right| + \left|\widetilde{\Psi}(s) - \Psi_0\right|\right) ds$$
$$\leq \mathcal{L}_2(\mathfrak{b} + \mathfrak{c})(\tau - \tau_0) \leq \mathcal{L}_2(\mathfrak{b} + \mathfrak{c})\hbar.$$
$$\left|\widetilde{\Psi}(\tau) - \Psi_2(\tau)\right| \leq \int_{\tau_0}^{\tau}\left|g(s, \widetilde{\Phi}(s), \widetilde{\Psi}(s)) - g(s, \Phi_1, \Psi_1)\right| ds$$
$$\leq \mathcal{L}_2 \int_{\tau_0}^{\tau} \left(\left|\widetilde{\Phi}(s) - \Phi_1\right| + \left|\widetilde{\Psi}(s) - \Psi_1\right|\right) ds$$
$$\leq \mathcal{L}_2 \int_{\tau_0}^{\tau} \left(\mathcal{L}_1(\mathfrak{b} + \mathfrak{c})\hbar + \mathcal{L}_2(\mathfrak{b} + \mathfrak{c})\hbar\right) ds$$
$$\leq \frac{\mathcal{L}_1\mathcal{L}_2(\mathfrak{b} + \mathfrak{c})\hbar^2}{2} + \frac{\mathcal{L}_2^{\,2}(\mathfrak{b} + \mathfrak{c})\hbar^2}{2}$$
$$\frac{\mathcal{L}_2(\mathcal{L}_1 + \mathcal{L}_2)(\mathfrak{b} + \mathfrak{c})\hbar^2}{2}.$$

$$u_n = \frac{\mathcal{L}_2(\mathcal{L}_1 + \mathcal{L}_2)^{n-1}(\mathfrak{b} + \mathfrak{c})\hbar^n}{n!}.$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{\mathcal{L}_2(\mathcal{L}_1 + \mathcal{L}_2)^n(\mathfrak{b} + \mathfrak{c})\hbar^{n+1} / (n+1)!}{\mathcal{L}_2(\mathcal{L}_1 + \mathcal{L}_2)^{n-1}(\mathfrak{b} + \mathfrak{c})\hbar^n / n!}\right)$$
$$= \lim_{n\to\infty} \frac{(\mathcal{L}_1 + \mathcal{L}_2)\hbar}{n} = 0 < 1.$$

$(n - 1)$ is replaced by n. When $n = 1$, we have :

$$\left|\widetilde{\Phi}(\tau) - \Phi_n(\tau)\right| \leq \int_{\tau_0}^{\tau}\left|\mathcal{F}(s, \widetilde{\Phi}(s), \widetilde{\Psi}(s)) - \mathcal{F}(s, \Phi_{n-1}, \Psi_{n-1})\right| ds$$
$$\leq \mathcal{L}_1 \int_{\tau_0}^{\tau} \left(\left|\widetilde{\Phi}(s) - \Phi_{n-1}\right| + \left|\widetilde{\Psi}(s) - \Psi_{n-1}\right|\right) ds$$

$$\leq \mathcal{L}_1\Big(\int_{\tau_0}^{\tau}\frac{\mathcal{L}_1(\mathcal{L}_1+\mathcal{L}_2)^{n-2}(\mathfrak{b}+\mathfrak{c})(s-\tau_0)^{n-1}}{(n-1)!}ds + \int_{\tau_0}^{\tau}\frac{\mathcal{L}_2(\mathcal{L}_1+\mathcal{L}_2)^{n-2}(\mathfrak{b}+\mathfrak{c})(s-\tau_0)^{n-1}}{(n-1)!}ds\Big)$$

$$\leq \int_{\tau_0}^{\tau}\frac{\mathcal{L}_1^{2}(\mathcal{L}_1+\mathcal{L}_2)^{n-2}(\mathfrak{b}+\mathfrak{c})(s-\tau_0)^{n-1}}{(n-1)!}ds + \int_{\tau_0}^{\tau}\frac{\mathcal{L}_1\mathcal{L}_2(\mathcal{L}_1+\mathcal{L}_2)^{n-2}(\mathfrak{b}+\mathfrak{c})(s-\tau_0)^{n-1}}{(n-1)!}ds)$$

$$\leq \frac{\mathcal{L}_1^{2}(\mathcal{L}_1+\mathcal{L}_2)^{n-2}(\mathfrak{b}+\mathfrak{c})}{(n-1)!}\left[\frac{(s-\tau_0)}{n}\right]_{\tau_0}^{\tau} + \frac{\mathcal{L}_1\mathcal{L}_2(\mathcal{L}_1+\mathcal{L}_2)^{n-2}(\mathfrak{b}+\mathfrak{c})}{(n-1)!}\left[\frac{(s-\tau_0)}{n}\right]_{\tau_0}^{\tau}$$

$$\leq \frac{\mathcal{L}_1^{2}(\mathcal{L}_1+\mathcal{L}_2)^{n-2}(\mathfrak{b}+\mathfrak{c})(\tau-\tau_0)^{n}}{n!} + \frac{\mathcal{L}_1\mathcal{L}_2(\mathcal{L}_1+\mathcal{L}_2)^{n-2}(\mathfrak{b}+\mathfrak{c})(\tau-\tau_0)^{n}}{n!}$$

$$\leq \frac{\mathcal{L}_1(\mathcal{L}_1+\mathcal{L}_2)(\mathcal{L}_1+\mathcal{L}_2)^{n-2}(\mathfrak{b}+\mathfrak{c})(\tau-\tau_0)^{n}}{n!}$$

$$\leq \frac{\mathcal{L}_1(\mathcal{L}_1+\mathcal{L}_2)^{n-1}(\mathfrak{b}+\mathfrak{c})(\tau-\tau_0)^{n}}{n!}$$

$$\leq \frac{\mathcal{L}_1(\mathcal{L}_1+\mathcal{L}_2)^{n-1}(\mathfrak{b}+\mathfrak{c})^{n}\hbar^{n}}{n!}.$$

Similarly:

$$\left|\widetilde{\Psi}(\tau)-\Psi_n(\tau)\right| \leq \int_{\tau_0}^{\tau}\left|g(s,\widetilde{\Phi}(s),\widetilde{\Psi}(s)) - g(s,\Phi_{n-1},\Psi_{n-1})\right|ds$$

$$\leq \mathcal{L}_2\int_{\tau_0}^{\tau}\left(\left|\widetilde{\Phi}(s)-\Phi_{n-1}\right| + \left|\widetilde{\Psi}(s)-\Psi_{n-1}\right|\right)ds$$

$$\leq \mathcal{L}_2\Big(\int_{\tau_0}^{\tau}\frac{\mathcal{L}_1(\mathcal{L}_1+\mathcal{L}_2)^{n-2}(\mathfrak{b}+\mathfrak{c})(s-\tau_0)^{n-1}}{(n-1)!}ds + \int_{\tau_0}^{\tau}\frac{\mathcal{L}_2(\mathcal{L}_1+\mathcal{L}_2)^{n-2}(\mathfrak{b}+\mathfrak{c})(s-\tau_0)^{n-1}}{(n-1)!}ds\Big)$$

$$\leq \int_{\tau_0}^{\tau}\frac{\mathcal{L}_1\mathcal{L}_2(\mathcal{L}_1+\mathcal{L}_2)^{n-2}(\mathfrak{b}+\mathfrak{c})(s-\tau_0)^{n-1}}{(n-1)!}ds + \int_{\tau_0}^{\tau}\frac{\mathcal{L}_2^{2}(\mathcal{L}_1+\mathcal{L}_2)^{n-2}(\mathfrak{b}+\mathfrak{c})(s-\tau_0)^{n-1}}{(n-1)!}ds)$$

$$\leq \frac{\mathcal{L}_1\mathcal{L}_2(\mathcal{L}_1+\mathcal{L}_2)^{n-2}(\mathfrak{b}+\mathfrak{c})}{(n-1)!}\left[\frac{(s-\tau_0)}{n}\right]_{\tau_0}^{\tau} + \frac{\mathcal{L}_2^{2}(\mathcal{L}_1+\mathcal{L}_2)^{n-2}(\mathfrak{b}+\mathfrak{c})}{(n-1)!}\left[\frac{(s-\tau_0)}{n}\right]_{\tau_0}^{\tau}$$

$$\leq \frac{\mathcal{L}_1\mathcal{L}_2(\mathcal{L}_1+\mathcal{L}_2)^{n-2}(\mathfrak{b}+\mathfrak{c})(\tau-\tau_0)^{n}}{n!} + \frac{\mathcal{L}_2^{2}(\mathcal{L}_1+\mathcal{L}_2)^{n-2}(\mathfrak{b}+\mathfrak{c})(\tau-\tau_0)^{n}}{n!}$$

$$\leq \frac{\mathcal{L}_2(\mathcal{L}_1+\mathcal{L}_2)(\mathcal{L}_1+\mathcal{L}_2)^{n-2}(\mathfrak{b}+\mathfrak{c})(\tau-\tau_0)^{n}}{n!}$$

$$\leq \frac{\mathcal{L}_2(\mathcal{L}_1+\mathcal{L}_2)^{n-1}(\mathfrak{b}+\mathfrak{c})(\tau-\tau_0)^{n}}{n!}$$

$$\leq \frac{\mathcal{L}_2(\mathcal{L}_1+\mathcal{L}_2)^{n-1}(\mathfrak{b}+\mathfrak{c})^{n}\hbar^{n}}{n!}.$$

For $n = 1$. Thus, by induction, the inequality holds for all (n) on $[\tau_0,\tau_0+\hbar]$. Hence, we have:

$$\left|\widetilde{\Phi}(\tau)-\Phi_n(\tau)\right| \leq \frac{\mathcal{L}_1(\mathcal{L}_1+\mathcal{L}_2)^{n-1}(\mathfrak{b}+\mathfrak{c})^{n}\hbar^{n}}{n!},$$

$$\left|\widetilde{\Psi}(\tau)-\Psi_n(\tau)\right| \leq \frac{\mathcal{L}_2(\mathcal{L}_1+\mathcal{L}_2)^{n-1}(\mathfrak{b}+\mathfrak{c})^{n}\hbar^{n}}{n!}.$$

for $n = 1,2,3,\ldots$ on $[\tau_0,\tau_0+\hbar]$.

Now the series$\left(\sum_{n=0}^{\infty}\frac{\mathcal{L}_1(\mathcal{L}_1+\mathcal{L}_2)^{n-1}(\mathfrak{b}+\mathfrak{c})^{n}\hbar^{n}}{n!}\right)$and $\left(\sum_{n=0}^{\infty}\frac{\mathcal{L}_2(\mathcal{L}_1+\mathcal{L}_2)^{n-1}(\mathfrak{b}+\mathfrak{c})^{n}\hbar^{n}}{n!}\right)$converges.

Thus $\left(\lim_{n\to\infty}\frac{\mathcal{L}_1(\mathcal{L}_1+\mathcal{L}_2)^{n-1}(\mathfrak{b}+\mathfrak{c})^{n}\hbar^{n}}{n!}=0\right)$ and $\left(\lim_{n\to\infty}\frac{\mathcal{L}_2(\mathcal{L}_1+\mathcal{L}_2)^{n-1}(\mathfrak{b}+\mathfrak{c})^{n}\hbar^{n}}{n!}=0\right)$. So, $(\widetilde{\Phi}(\tau) = \lim_{n\to\infty}\Phi_n(\tau))$ and $(\widetilde{\Psi}(\tau) = \lim_{n\to\infty}\Psi_n(\tau))$ on $[\tau_0,\tau_0+\hbar]$. But recall that $(\Phi(\tau) = \lim_{n\to\infty}\Phi_n(\tau))$ and $(\Psi(\tau) = \lim_{n\to\infty}\Psi_n(\tau))$ on this interval. Thus,

$\widetilde{\Phi}(\tau) = \Phi(\tau)$,

$\widetilde{\Psi}(\tau) = \Psi(\tau)$.

on $[\tau_0,\tau_0+\hbar]$. Thus the solution (Φ,Ψ) of the basic initial-value problem is unique on $[\tau_0,\tau_0+\hbar]$.

We have thus proved that the basic initial-value problem has a unique solution on $[\tau_0,\tau_0+\hbar]$. As we pointed out at the start of the proof, we can carry through similar arguments on the interval $[\tau_0,\tau_0+\hbar]$. We thus conclude that differential equations $(\frac{d\Phi}{d\tau} = \mathcal{F}(\tau,\Phi,\Psi)$ and $\frac{d\Psi}{d\tau} = g(\tau,\Phi,\Psi))$ have a unique solution (Φ,Ψ) such that $(\Phi(\tau_0) = \Phi_0$ and $\Psi(\tau_0) = \Psi_0)$ on $|\tau-\tau_0| \leq \hbar$. ■

Theorem 4.2: In system (1.1), if the functions $\mathcal{F}$ and g are continuous and satisfy lipschiz condition, then the system (1.1) is Ulam stability.

Proof: The system of equations:

$$\Phi(\tau) = \Phi_0 + \int_{\tau_0}^{\tau}\mathcal{F}(s,\Phi(s),\Psi(s))ds,$$

$$\Psi(\tau) = \Psi_0 + \int_{\tau_0}^{\tau}g(s,\Phi(s),\Psi(s))ds,$$

has Ulam stability if for any approximate solution $(\widetilde{\Phi}(\tau),\widetilde{\Psi}(\tau))$ that satifies the system within some small error bound δ, there exists an exact solution $(\Phi(\tau),\Psi(\tau))$ such that

$$\sup\left(\left\|\widetilde{\Phi}(\tau)-\Phi(\tau)\right\| + \left\|\widetilde{\Psi}(\tau)-\Psi(\tau)\right\| \leq \epsilon,\ \tau\in[\tau_0,\tau]\right.$$

where $\epsilon > 0$ depends on $\delta > 0$.

Let $(\widetilde{\Phi}(\tau),\widetilde{\Psi}(\tau))$ be an approximate solution such that :

$$\left\|\widetilde{\Phi}(\tau)-\Phi_0-\int_{\tau_0}^{\tau}\mathcal{F}(s,\widetilde{\Phi}(s),\widetilde{\Psi}(s))ds\right\| \leq \delta,$$

$$\left\|\widetilde{\Psi}(\tau)-\Psi_0-\int_{\tau_0}^{\tau}g(s,\widetilde{\Phi}(s),\widetilde{\Psi}(s))ds\right\| \leq \delta,$$

For some small $\delta > 0$.

$$e_{\Phi}(\tau) = \widetilde{\Phi}(\tau)-\Phi(\tau),$$

$$e_{\Psi}(\tau) = \widetilde{\Psi}(\tau)-\Psi(\tau).$$

Then:

$\|e_\Phi(\tau)\| = \left\| \Phi_0 + \int_{\tau_0}^{\tau} \mathcal{F}(s, \widetilde{\Phi}(s), \widetilde{\Psi}(s)) ds - \Phi_0 - \int_{\tau_0}^{\tau} \mathcal{F}(s, \Phi(s), \Psi(s)) ds \right\|$

$$\leq \int_{\tau_0}^{\tau} \left\| \mathcal{F}(s, \widetilde{\Phi}(s), \widetilde{\Psi}(s)) - \mathcal{F}(s, \Phi(s), \Psi(s)) \right\| ds$$

$$\leq \int_{\tau_0}^{\tau} \mathcal{L} \left\| (\widetilde{\Phi} - \Phi) + (\widetilde{\Psi} - \Psi) \right\| ds$$

$$\leq \int_{\tau_0}^{\tau} \mathcal{L}(\|e_\Phi(s)\| + \|e_\Psi(s)\|) ds + \delta.$$

Similarly:

$$\|e_\Psi(\tau)\| \leq \int_{\tau_0}^{\tau} \mathcal{L}(\|e_\Phi(s)\| + \|e_\Psi(s)\|) ds + \delta.$$

Let $E(\tau) = \|e_\Phi(\tau)\| + \|e_\Psi(\tau)\|$, then:

$$E(\tau) \leq \int_{\tau_0}^{\tau} \mathcal{L}\, E(s) ds + 2\delta.$$

Applying Gronwall's inequality:

$E(\tau) \leq 2\delta e^{\mathcal{L}(\tau - \tau_0)}$,

Thus, for any $\delta > 0$, the error between the approximate solution $(\widetilde{\Phi}(\tau), \widetilde{\Psi}(\tau))$ and the exact solution $(\Phi(\tau), \Psi(\tau))$ is bounded by: $\epsilon = 2\delta e^{\mathcal{L}(\tau - \tau_0)}$, thus the system is Ulam stability. ■

Ulam-Hyers-Rassias Stability: A system is said to have Ulam-Hyers-Rassias stability if, for any functions $\widetilde{\Phi}(\tau)$ and $\widetilde{\Psi}(\tau)$ that approximately satisfy the integral equations within a controlled deviation, there exist exact solutions $\Phi(\tau)$ and $\Psi(\tau)$ such that the deviations between the approximate and exact solutions are bounded by a function of the initial deviation.

Formally, if there exist functions $\epsilon_\Phi(\tau)$ and $\epsilon_\Psi(\tau)$ such that:

$$\left| \widetilde{\Phi}(\tau) - \Phi_0 - \int_{\tau_0}^{\tau} \mathcal{F}(s, \widetilde{\Phi}(s), \widetilde{\Psi}(s)) ds \right| \leq \epsilon_\Phi(\tau),$$

$$\left| \widetilde{\Psi}(\tau) - \Psi_0 - \int_{\tau_0}^{\tau} g(s, \widetilde{\Phi}(s), \widetilde{\Psi}(s)) ds \right| \leq \epsilon_\Psi(\tau),$$

then there exist exact solutions $(\Phi(\tau), \Psi(\tau))$ such that:

$$\left| \widetilde{\Phi}(\tau) - \Phi(\tau) \right| \leq \emptyset_{\Phi,\Psi}\big(\epsilon_\Phi(\tau)\big), \big(\epsilon_\Psi(\tau)\big),$$

$$\left| \widetilde{\Psi}(\tau) - \Psi(\tau) \right| \leq \emptyset_{\Phi,\Psi}\big(\epsilon_\Phi(\tau)\big), \big(\epsilon_\Psi(\tau)\big),$$

where $(\emptyset_\Phi, \emptyset_\Psi)$ is functions that depends on $\epsilon_\Phi(\tau)$ and $\epsilon_\Psi(\tau)$, respectively.

Proof: since ($\mathcal{F}$ and g) are satisfy lipschitz condition, there exist constans $\mathcal{L}_\mathcal{F}$ and $\mathcal{L}_g$ such that for all $\tau \in [\tau_0, \tau]$.

$|\mathcal{F}(\tau, \Phi_1, \Psi_1) - \mathcal{F}(\tau, \Phi_2, \Psi_2)| \leq \mathcal{L}_\mathcal{F}(|\Phi_1 - \Phi_2| + |\Psi_1 - \Psi_2|)$,

$|g(\tau, \Phi_1, \Psi_1) - g(\tau, \Phi_2, \Psi_2)| \leq \mathcal{L}_g(|\Phi_1 - \Phi_2| + |\Psi_1 - \Psi_2|)$.

Let ($\epsilon_\Phi(\tau)$ and $\epsilon_\Psi(\tau)$) be the error functions between the approximate and exact solutions:

$$e_\Phi(\tau) = \widetilde{\Phi}(\tau) - \Phi(\tau),$$

$$e_\Psi(\tau) = \widetilde{\Psi}(\tau) - \Psi(\tau).$$

$$\widetilde{\Phi}(\tau) = \epsilon_\Phi(\tau) + \Phi_0 + \int_{\tau_0}^{\tau} \mathcal{F}(s, \widetilde{\Phi}(s), \widetilde{\Psi}(s)) ds,$$

$$\widetilde{\Psi}(\tau) = \epsilon_\Psi(\tau) + \Psi_0 + \int_{\tau_0}^{\tau} g\left(s, \widetilde{\Phi}(s), \widetilde{\Psi}(s)\right) ds,$$

Thus:

$\left| \widetilde{\Phi}(\tau) - \Phi(\tau) \right| \leq \epsilon_\Phi(\tau) + \Phi_0 + \int_{\tau_0}^{\tau} \mathcal{F}(s, \widetilde{\Phi}(s), \widetilde{\Psi}(s)) ds - \Phi_0 - \int_{\tau_0}^{\tau} \mathcal{F}(s, \Phi(s), \Psi(s)) ds$

$$\leq \epsilon_\Phi(\tau) + \int_{\tau_0}^{\tau} \left| \mathcal{F}(s, \widetilde{\Phi}(s), \widetilde{\Psi}(s)) - \mathcal{F}(s, \Phi(s), \Psi(s) \right| ds$$

$$\leq \epsilon_\Phi(\tau) + \int_{\tau_0}^{\tau} \mathcal{L}_\mathcal{F}\left(\left| \widetilde{\Phi} - \Phi \right| + \left| \widetilde{\Psi} - \Psi \right| \right) ds$$

$$|e_\Phi(\tau)| \leq \epsilon_\Phi(\tau) + \int_{\tau_0}^{\tau} \mathcal{L}_\mathcal{F}(|e_\Phi(s)| + |e_\Psi(s)|) ds .$$

Similarly:

$$|e_\Psi(\tau)| \leq \epsilon_\Psi(\tau) + \int_{\tau_0}^{\tau} \mathcal{L}_g(|e_\Phi(s)| + |e_\Psi(s)|) ds.$$

Let $E(\tau) = |e_\Phi(\tau)| + |e_\Psi(\tau)|$.

$$\Longrightarrow \quad E(\tau) \leq \epsilon_\Phi(\tau) + \epsilon_\Psi(\tau) + \int_{\tau_0}^{\tau} (\mathcal{L}_\mathcal{F} + \mathcal{L}_g) E(s) ds.$$

By Gronwall's inequality:

$$E(\tau) \leq \big(\epsilon_\Phi(\tau) + \epsilon_\Psi(\tau)\big) e^{\int_{\tau_0}^{\tau} (\mathcal{L}_\mathcal{F} + \mathcal{L}_g) ds}.$$

Thus,

$$E(\tau) \leq \big(\epsilon_\Phi(\tau) + \epsilon_\Psi(\tau)\big) e^{(\mathcal{L}_\mathcal{F} + \mathcal{L}_g)(\tau - \tau_0)}.$$

and hence,

$$e_\Phi(\tau) \leq E(\tau), \quad e_\Psi(\tau) \leq E(\tau),$$

This implies that:

$$e_\Phi(\tau) \leq \big(\epsilon_\Phi(\tau) + \epsilon_\Psi(\tau)\big) e^{(\mathcal{L}_\mathcal{F} + \mathcal{L}_g)(\tau - \tau_0)},$$

$$e_\Psi(\tau) \leq \big(\epsilon_\Phi(\tau) + \epsilon_\Psi(\tau)\big) e^{(\mathcal{L}_\mathcal{F} + \mathcal{L}_g)(\tau - \tau_0)}.$$

which further leads to

$$e_\Phi(\tau) \leq \emptyset_{\Phi,\Psi}\big(\epsilon_\Phi(\tau)\big), \big(\epsilon_\Psi(\tau)\big),$$

$$e_\Psi(\tau) \leq \emptyset_{\Phi,\Psi}\big(\epsilon_\Phi(\tau)\big), \big(\epsilon_\Psi(\tau)\big). ■$$

5. Graphs analysis

These graphs mathematically represent the dynamic interaction between society, $\Phi(\tau)$, and corruption, $\Psi(\tau)$, within the framework of the Lotka-Volterra model:

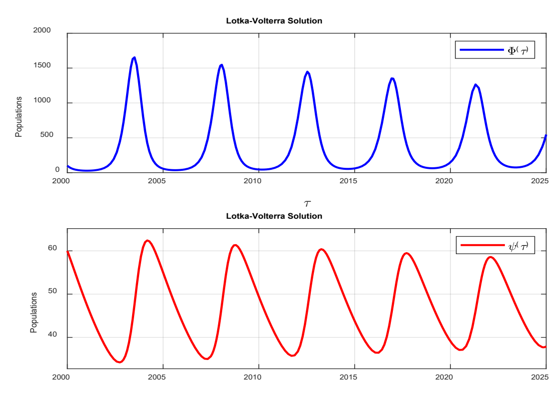

Figure 1: The damped oscillatory dynamics of the extended Lotka-Volterra model are illustrated through

the numerical solution over the interval $\tau \in [2000, 2025]$, using parameters $\xi_1 = 9.5$, $\xi_2 = 0.2$, $\alpha = 1 \times 10^{-4}$, $\rho_1 = 0.25$, $\rho_2 = 6.25 \times 10^{-4}$, and $\beta = 0.1$, with initial conditions $\Phi(0) = 100$ and $\Psi(0) = 60$.

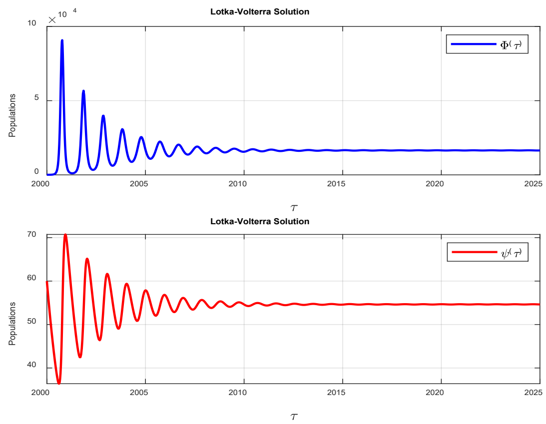

Figure 2: Transition to Stability via Damped Oscillations Simulation with $\xi_1 = 50$, $\xi_2 = 0.9$, $\alpha = 5 \times 10^{-5}$, $\rho_1 = 0.9$, $\rho_2 = 5.5 \times 10^{-5}$, and $\beta = 0.1$, and initial conditions $\Phi(0) = 100$ and $\Psi(0) = 60$.

In **Figure 1**, the top subplot (blue curve) depicts the temporal evolution of the societal state variable $\Phi(\tau)$, which exhibits damped oscillatory behavior. The amplitude of these oscillations gradually diminishes over time due to the nonlinear damping term $-\alpha\Phi^2(\tau)$, which prevents unbounded growth and produces successively lower peaks. The bottom subplot (red curve) shows the evolution of the corruption level $\Psi(\tau)$, which also demonstrates oscillatory dynamics but with lower amplitude and slower variation, influenced by the logarithmic damping term $-\beta\, ln\,(\Psi(\tau))$. The simulation highlights the adaptive interplay between societal response and corruption: while the initial societal resistance to increasing corruption is weak, a marked reaction emerges once corruption exceeds a critical threshold. This delayed yet intensified response suggests the presence of a nonlinear feedback mechanism, wherein societal resilience strengthens only after a tolerable limit has been surpassed.

In **Figure 2**, in contrast to the dynamics presented in **Figure 1**, this configuration induces more rapid and sharper oscillations in the societal state $\Phi(\tau)$ (blue curve) due to the larger influence of ξ_1 and ξ_2. The relatively small damping coefficient α permits these high-amplitude oscillations to persist longer, delaying convergence. Meanwhile, the corruption level $\Psi(\tau)$ (red curve) experiences sharp drops influenced by the dominant ρ_1 term, but its recovery is impeded by the small value of ρ_2. The logarithmic damping $-\beta\, ln\,(\Psi(\tau))$ further moderates its fluctuations. Overall, the simulation demonstrates how parameter tuning can lead the system from an initially unstable state toward a bounded and recurrent regime, highlighting the role of nonlinear damping in achieving long-term equilibrium.

Conclusion

In this study, we have established that the functions governing the dynamics of corruption and society satisfy the Lipschitz condition under given assumptions, ensuring the existence and uniqueness of a solution via the Picard approximation theorem.

Our results confirm that corruption and society evolve in a predictable manner over time, with their interactions leading to a stable equilibrium. While corruption cannot be entirely eradicated, it can be controlled and maintained at a manageable level, preventing extreme fluctuations.

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Conflict of interest

None.

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