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Generalization of the Transformation Method to the Stratonovich Formula for Solving Stochastic Differential Equations

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Abstract

In this research, the reducible method or what is called the transformation method was generalized to the Stratonovich formula used to solve stochastic differential equations (SDEs), and the general formulas for the solutions and their theories were reached and the conditions necessary for reducing the stochastic differential equation were clarified by generalizing this method from the Ito integration formula to the Stratonovich integration formula and the transformation method between them, and these two integration formulas (Ito formula and Stratonovich formula) were applied to a group of diverse examples and the solutions were obtained and drawn (by MATLAB program) and the results of the solutions for both methods were compared.

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1. Introduction

The methods of solving stochastic differential equations (SDEs) are limited and not comprehensive for all types. This is attributed to the nature of the stochastic term in the SDEs. There are many methods for finding solutions to stochastic equations according to the nature of the equation, either directly, such as linear SDEs, which have a general formula for the solution, or stochastic equations may be non-linear but reducible to linearity, such as the reduction method that we are dealing with in this research. Several previous studies have addressed the topic of solving SDEs, including a study by Fadel and Sobhi (2016) to find the solution for stochastic equations using the method of arranging the variable for the Stratonovich formula. There are also several studies that have addressed numerical methods in finding solutions to SDEs, including Rosa (2016), Dhanpal (2021), Bayram et al. (2018). There is also a study conducted by Salem and Abdel Rahman (2023) that dealt with finding the analytical solution to the stochastic equations of Ito's formula [1],[2].

1- 1: Wiener Process

Let $\{w(t): t \geq 0\}$ is a stochastic process on $[0, T]$, it is called a Wiener process if

1. $w(0) = 0$
2. $l, v \geq 0 \implies w(l) - w(v) \sim N(0, |l - v|)$
3. $0 \leq v, t, s, k \leq T \implies w(t) - w(v)$ & $w(k) - w(s)$ are independent increment [3].

1-2: Stochastic Differential Equations (SDEs)

The SDEs consists of two main parts, the first is represented by the Riemann integral and the second part is represented by the stochastic integral, and thus the following equation

$$dy(t) = \hbar(y(t))dt + g(y(t))dw(t) \quad \ldots\ldots\ldots (1)$$

is called a SDEs, where $\hbar(y(t))$, $g(y(t))$ are nonlinear deterministic functions and $dw(t)$ represents the differentiation of the Wiener process $w(t)$ [3].

The general form of the linear SDE is

$$dy(t) = [\hbar_1(t)y + \hbar_2(t)]dt + [g_1(t)y + g_2(t)]dw(t) \quad (2)$$

,and the general form of its analytical solution is

$$y(t) = M_{t,t_o}\left[y_o + \int_{t_o}^{t} M_{s,t_o}^{-1}[\hbar_2(s) - g_1(s)]ds + \int_{t_o}^{t} M_{s,t_o}^{-1} g_2(s)dw(s)\right] \quad (3)$$

Where

$$M_{t,t_o} = e^{\left(\int_{t_o}^{t}\left[\hbar_1(s) - \frac{1}{2}{g_1}^2(s)\right]ds + \int_{t_o}^{t} g_1(s)dw(s)\right)}$$

[3] and [4].

1-3: Ito Integration

The stochastic Ito integral is defined as

$$\int_0^T \beta(t, w(t))\, dw(t) = \lim_{n\to\infty}\sum_{i=1}^{n} \beta(t_i^{(n)}, w)\left(w(t_{i+1}^{(n)}) - w(t_i^{(n)})\right)$$

this integration is called the stochastic Ito integral. [4]

1-4: Stratonovich Integration

The Stratonovich integral is defined as

$$\int_0^T \beta(t, w(t)) \circ dw(t) = \lim_{n\to\infty}\sum_{i=1}^{n} \frac{1}{2}\left(\beta(t_i^{(n)}, w) + \beta(t_{i+1}^{(n)}, w)\right)\left(w(t_{i+1}^{(n)}) - w(t_i^{(n)})\right)$$

We notice that this integral takes the middle of the intervals $[t_i^{(n)}, t_{i+1}^{(n)}]$ while the Ito integral takes the beginning of these intervals.

To distinguish the Stratonovich integral from the Ito integral, the following sign "∘" is placed in the Stratonovich integral , while the Ito integral is without that sign [5].

1-5: The Exact Solution (The Reducible Method)

The methods of solving SDEs are characterized by being diverse and multiple due to the nature of the stochastic term in the equation. The equations may be linear and thus have a general formula for the solution. In most cases, the equations are non-linear, so we resort to solving them using one of the analytical methods if possible, or we turn to numerical solution methods. We mention some analytical methods for solving rates: the reduction method, the integral factor method, and the method of equations with a direct solution. Each method has conditions for its application in accordance with the stochastic equation.

1-6: The Reducible Method for Ito SDEs

The reduction method is one of the analytical methods for solving nonlinear SDEs (Ito/Stratonovich) provided that the equation is reducible by fulfilling the reduction conditions (mentioned above). In this way, we can convert the equation into a linear one. We can find the solution formula for it directly. We will explain this method for the Ito formula and then we will move on to generalizing the method and obtaining the general solution formulas for the Stratonovich formula.

Let we have the following nonlinear SDEs

$$dy(t) = \hbar(y(t))dt + g(y(t))dw(t)$$

We want to reduce it by the transformation function $x = U(y)$ to the linear equation.

That is, the eq.(1) has a solution $y(t)$ in the form $y = L(x)$ which is the inverse transformation $x = U(y)$

We use Ito's formula for the function $x = U(y(x))$ on eq.(1) and we get

$$dU = \left[U_y\hbar(y) + \frac{1}{2}U_{yy}g(y)\right]dt + \left[U_y g(y)\right]dw(t) \quad (4)$$

Now we equate eqs. (2) and (4) and we get

$$U_y\hbar(y) + \frac{1}{2}U_{yy}g^2(y) = \hbar_1 U(y) + \hbar_2 \quad (5)$$

$$U_y g(y) = g_1 U(y) + g_2 \quad (6)$$

We find U_y from equation (6), that is

$$U_y - \frac{g_1}{g(y)}U(y) = \frac{g_2}{g(y)} \quad (7)$$

This is a linear ordinary differential equation [6],[7] , and

$$\mathcal{M}(y) = \int_{y_0}^{y}\left(\frac{1}{g(s)}\right)ds$$

<u>**First case:**</u> If $g_1(t) = 0$ in eq.(7), then its solution is in the following form:

$$U(y) = g_2(t)\mathcal{M}(y) + c \quad (8)$$

<u>**Second case:**</u> If $g_1(t) \neq 0$ in eq.(7), and to solve this equation, we find its integration factor, which is

$$e^{\int_{y_0}^{y}\left(-\frac{g_1}{g(s)}\right)ds} = e^{-g_1\mathcal{M}(y)}$$

So the solution to eq.(7) is as follows:

$$U(y) = e^{g_1\mathcal{M}(y)} \int_{y_o}^{y} e^{-g_1\mathcal{M}(s)} \frac{g_2}{g(s)} ds + c\, e^{g_1\mathcal{M}(y)} \quad \dots\dots (9)$$

Where c is the constant of integration

$$U(y) = \left(\frac{-g_2}{g_1}\right) e^{g_1\mathcal{M}(y)} \int_{y_o}^{y} e^{-g_1\mathcal{M}(s)} \left(\frac{-g_1}{g(s)}\right) ds + c\, e^{g_1\mathcal{M}(y)}$$

$$U(y) = \left(\frac{-g_2}{g_1}\right) e^{g_1\mathcal{M}(y)} e^{-g_1\mathcal{M}(y)} + c\, e^{g_1\mathcal{M}(y)}$$

$$U(y) = c\, e^{g_1\mathcal{M}(y)} - \frac{g_2}{g_1} \quad \dots\dots\dots (10)$$

We find the derivatives U_y , U_{yy} from equation (10)

$$U_y = c g_1\, e^{g_1\mathcal{M}(y)}\, \mathcal{M}_y(y) = c\, e^{g_1\mathcal{M}(y)}\, \frac{g_1}{g(y)} \quad (11)$$

$$U_{yy} = c\, e^{g_1\mathcal{M}(y)} \left[-\frac{g_1 g_y(y)}{g^2(y)} + \frac{{g_1}^2}{g(y)} \frac{1}{g(y)}\right]$$

$$U_{yy} = c\, e^{g_1\mathcal{M}(y)} \left[-\frac{g_1 g_y(y)}{g^2(y)} + \frac{{g_1}^2}{g^2(y)}\right] \quad (12)$$

We substitute equations (10)-(12) in eq.(5) as follows

$$c\, e^{g_1\mathcal{M}(y)} \left[\frac{g_1}{g(y)} \hbar(y) + \frac{1}{2}\left(-\frac{g_1 g_y(y)}{g^2(y)} + \frac{{g_1}^2}{g^2(y)}\right) g^2(y)\right] = \hbar_1 \left[c\, e^{g_1\mathcal{M}(y)} - \frac{g_2}{g_1}\right] + \hbar_2$$

$$c\, e^{g_1\mathcal{M}(y)} \left[\frac{\hbar(y)}{g(y)} g_1 + \frac{1}{2}{g_1}^2 - \frac{1}{2} g_1 g_y(y)\right] = c\, e^{g_1\mathcal{M}(y)} \hbar_1 - \frac{\hbar_1 g_2}{g_1} + \hbar_2$$

$$c\, e^{g_1\mathcal{M}(y)} \left[\frac{\hbar(y)}{g(y)} g_1 + \frac{1}{2}{g_1}^2 - \frac{1}{2} g_1 g_y(y) - \hbar_1\right] = \hbar_2 - \frac{\hbar_1 g_2}{g_1}$$

Suppose that

$$\varphi(y) = \frac{\hbar(y)}{g(y)} - \frac{1}{2} g_y(y)$$

Thus we get

$$c\, e^{g_1\mathcal{M}(y)} \left[\varphi(y) g_1 + \frac{1}{2}{g_1}^2 - \hbar_1\right] = \hbar_2 - \frac{\hbar_1\, g_2}{g_1} \quad \dots\dots\dots (13)$$

We derive equation (13) with respect to y

$$c\, e^{g_1\mathcal{M}(y)} [g_1 \varphi_y] + \left[\varphi g_1 + \frac{1}{2}{g_1}^2 - \hbar_1\right] \left[c g_1\, e^{g_1\mathcal{M}(y)} \mathcal{M}_y(y)\right] = 0$$

$$c g_1\, e^{g_1\mathcal{M}(y)} \left[\varphi_y(y) + \frac{\varphi(y) g_1}{g(y)}\right] = \left[\hbar_1 - \frac{{g_1}^2}{2}\right] \frac{c g_1\, e^{g_1\mathcal{M}(y)}}{g(y)}$$

We get

$$g(y)\varphi_y(y) + \varphi(y) g_1 = \hbar_1 - \frac{{g_1}^2}{2}$$

$$\frac{d}{dy}\Big(g(y)\varphi_y(y)\Big) + g_1 \varphi_y(y) = 0$$

$$g_1 = \frac{-\frac{d}{dy}\Big(g(y)\varphi_y(y)\Big)}{\varphi_y(y)} \quad \dots\dots\dots (14)$$

The eq. (14) represents the condition for reducing from a nonlinear equation to a linear equation, i.e. if the value $g_1 = 0$ then the transformation function will be in the following form (note eq.(8))

$$U(y) = g_2\, \mathcal{M}(y) + c$$

If $g_1 \neq 0$, the transformation function will be as follows (note eq.(10)) [2] and [8].

$$U(y) = c\, e^{g_1\mathcal{M}(y)}$$

1-7: **Theorem (1):** "Reference [9]" The nonlinear stochastic differential eq.(1) can be reduced to a linear one if the following condition is met

$$\varphi_y(y) = 0 \quad \text{Or} \quad \frac{d}{dy}(g_1) = \frac{d}{dy}\left(\frac{-\frac{d}{dy}\big(g(y)\varphi_y(y)\big)}{\varphi_y(y)}\right) = 0$$

Where

$$\varphi(y) = \frac{\hbar(y)}{g(y)} - \frac{1}{2} g_y(y)$$

2. RESULTS:

2-1: The relationship between Ito and Stratonovich equations

We can transform the Stratonovich formula in the SDEs defined in equation

$$dy(t) = \hbar\big(y(t)\big)dt + g\big(y(t)\big) \circ dw(t)$$

into the Ito formula through the following relation

$$dy(t) = \left[\hbar(y(t), t) + \frac{1}{2} g(y(t), t) g_y(y(t), t)\right] dt + [g(y(t), t)] dw(t)$$

Thus, we have obtained a stochastic differential equation of the Ito type equivalent to the stochastic equation of the Stratonovich type [10] and [11].

2-2: The Reducible Method For Stratonovich SDEs

We have the following nonlinear stochastic equation in Stratonovich form

$$dy(t) = \hbar\big(y(t)\big)dt + g\big(y(t)\big) \circ dw(t) \quad (15)$$

It will be reduced by the transformation function x = U(y) to the linear equation.

Eq.(15) in Stratonovich form and is equivalent to the following stochastic equation in Ito form

$$dy(t) = \left(\hbar(y) + \frac{1}{2}g(y)g_y(y)\right)dt + g(y)dw(t) \quad (16)$$

The solution $y(t)$ is in the form $y = L(x)$ which is the inverse transformation $x = U(y)$

We substitute Ito's formula for the function $x = U(y)$ into eq.(16)

$$dU = \left[U_y\left(\hbar(y) + \frac{1}{2}g(y)g_y(y)\right) + \frac{1}{2}U_{yy}g(y)\right]dt + \left[U_y g(y)\right]dw(t) \quad (17)$$

We equate eq.(2) and (17)

$$U_y\left(\hbar(y) + \frac{1}{2}g(y)g_y(y)\right) + \frac{1}{2}U_{yy}g^2(y) = \hbar_1 U(y) + \hbar_2 \quad (18)$$

$$U_y g(y) = g_1 U(y) + g_2 \quad (19)$$

When we deduce the equations of the Stratonovich formula and find the special parameters to transform stochastic eq.(15) into the linear form as in eq.(2) in the same way that was followed for the previous Ito formula, we obtain the following equations:

Suppose that

$$\rho(y) = \frac{\hbar(y)}{g(y)}$$

Therefore

$$c\, e^{g_1\mathcal{M}(y)}\left[\rho(y)g_1 + \frac{1}{2}{g_1}^2 - \hbar_1\right] = \hbar_2 - \frac{\hbar_1\, g_2}{g_1} \quad (20)$$

$$g_1 = \frac{-\frac{d}{dy}\left(g(y)\rho_y(y)\right)}{\rho_y(y)} \quad (21)$$

2-3: **Theorem (2):** "Reference [7]" The stochastic equation in the nonlinear Stratonovich form can be reduced to linear if the following condition is met:

$$\rho_y(y) = 0 \quad \text{Or} \quad \frac{d}{dy}(g_1) = \frac{d}{dy}\left(\frac{-\frac{d}{dy}\left(g(y)\rho_y(y)\right)}{\rho_y(y)}\right) = 0$$

Where

$$\rho(y) = \frac{\hbar(y)}{g(y)}$$

3. Application

In this paragraph, we explain the method by some examples by using the two formulas (Ito's formula and Stratonovich's formula):

Example (1): Let $y(t)$ be a solution to the following nonlinear equation:

$$dy(t) = y^4 dt + y dw \quad , \quad y(0) = 1 \quad \ldots\ldots\ldots(22)$$

Solution:- (Using Ito formula)

From eq.(1)

$$\hbar(y) = y^4 \quad , \quad g(y) = y$$

To show that eq.(22) reducible or not:

$$\varphi(y) = \frac{\hbar(y)}{g(y)} - \frac{1}{2}g_y(y) = \frac{y^4}{y} - \frac{1}{2}(1) = y^3 - \frac{1}{2}$$

$$\Rightarrow \quad \varphi_y(y) = 3y^2$$

$$\frac{d}{dy}(g_1) = \frac{\partial}{\partial y}\left(\frac{-\frac{d}{dy}\left(g(y)\varphi_y(y)\right)}{\varphi_y(y)}\right) = \frac{d}{dy}\left(\frac{-\frac{d}{dy}(3y^3)}{3y^2}\right) = 0$$

According to theorem (1), the stochastic equation is reducible.

Now we will reduce the nonlinear stochastic equation to a linear one, as follows From eq.(14), we get

$$g_1 = \frac{-\frac{d}{dy}\left(g(y)\varphi_y(y)\right)}{\varphi_y(y)} = \frac{-9y^2}{3y^2} = -3$$

Let $g_2 = 0$, apply equation (10)

$$U(y) = c\, e^{g_1\mathcal{M}(y)} - \frac{g_2}{g_1} = e^{-3ln\left(\frac{y}{y_0}\right)} = \frac{1}{y^3} \quad , (c = 1)$$

Where

$$\mathcal{M}(y) = \int_{y_0}^{y}\left(\frac{1}{g(s)}\right)ds = \int_{1}^{y}\left(\frac{1}{s}\right)ds = ln(y)$$

Applying eq.(13), we get

$$e^{-3ln(y)}\left[\left(y^3 - \frac{1}{2}\right)(-3) + \frac{1}{2}(-3)^2 - \hbar_1\right] = \hbar_2 - \frac{(\hbar_1)(0)}{(-3)}$$

$$\frac{1}{y^3}(-3y^3 + 6 - \hbar_1) = \hbar_2$$

$$-3 + \frac{6}{y^3} = \hbar_2 + \frac{\hbar_1}{y^3} \quad \Rightarrow \quad \hbar_1 = 6 \quad , \quad \hbar_2 = -3$$

So we substitute the values $\hbar_1 = 6$, $\hbar_2 = -3$, $g_1 = -3$, $g_2 = 0$ into eq.(2) we get

$$dx(t) = [6x - 3]dt + [-3x]dw(t)$$

Now the nonlinear SDE. in Ito form has been reduced to a linear equation, and its solution is as follows (according eq.(3))

$$M_{t,t_o} = e^{\left(\int_0^t\left[6 - \frac{1}{2}(-3)^2\right]ds + \int_0^t -3dw(s)\right)} = e^{\frac{3}{2}t - 3w(t)}$$

$$x(t) = e^{\frac{3}{2}t-3w(t)}\left[x_o + \int_0^t e^{-\frac{3}{2}s+3w(s)}(-3{x_0}^3+3)ds\right]$$

$$x(t) = e^{\frac{3}{2}t-3w(t)}\left[x_o - 2(-{x_0}^3+1)e^{-\frac{3}{2}t+3w(t)} + 2(-{x_0}^3 + 1)\right]$$

$$x(t) = (x_o - 2{x_0}^3+2)e^{\frac{3}{2}t-3w(t)} + 2{x_0}^3 - 2$$

Since the solution $y(t)$ is the inverse of the transformation $x = U(y) = \frac{1}{y^3}$

And $x_o = x(0) = U\big(y(0)\big) = \frac{1}{y(0)^3} = \frac{1}{y(0)^3} = 1$, then

$$y(t) = \frac{1}{\sqrt[3]{e^{\frac{3}{2}t-3w(t)}}}$$

Solution:- (Using Stratonovich formula)

To explain if eq.(22) reducible or not by using Stratonovich formula

Here

$$\rho(y) = \frac{\hbar(y)}{g(y)} = \frac{y^4}{y} = y^3 \quad \Longrightarrow \quad \rho_y(y) = 3y^2$$

$$\frac{\partial}{\partial y}(g_1) = \frac{\partial}{\partial y}\left(\frac{-\frac{d}{dy}\big(g(y)\rho_y(y)\big)}{\rho_y(y)}\right) = \frac{\partial}{\partial y}\left(\frac{-\frac{d}{dy}(3y^3)}{3y^2}\right) = 0$$

According to Theorem (2), the stochastic equation in Stratonovich form is reducible.

To reduce the eq.(22) to linear: From eq.(21) we can see

$$g_1 = \frac{-\frac{d}{dy}\big(g(y)\rho_y(y)\big)}{\rho_y(y)} = \frac{-\frac{d}{dy}(3y^3)}{3y^2} = -3$$

Let $g_2 = 0$, and

$$U(y) = c\, e^{g_1\mathcal{M}(y)} - \frac{g_2}{g_1} = e^{-3ln\left(\frac{y}{y_0}\right)} = \frac{1}{y^3} \quad , \quad (c=1)$$

Where

$$\mathcal{M}(y) = \int_{y_0}^{y}\left(\frac{1}{g(s)}\right)ds = \int_1^y\left(\frac{1}{s}\right)ds = ln(y)$$

Applying eq.(20), we get

$$e^{-3ln(y)}\left[(y^3)(-3) + \frac{1}{2}(-3)^2 - \hbar_1\right] = \hbar_2$$

$$\frac{1}{y^3}\left(-3y^3 + \frac{9}{2} - \hbar_1\right) = \hbar_2$$

$$-3 + \frac{1}{y^3}\left(\frac{9}{2} - \hbar_1\right) = \hbar_2 \quad \Longrightarrow \hbar_1 = \frac{9}{2} \;,\; \hbar_2 = -3$$

So we substitute the values $\hbar_1 = \frac{9}{2}$, $\hbar_2 = -3$, $g_1 = -3$, $g_2 = 0$ in eq.(2)

$$dx(t) = \left[\frac{9}{2}x - 3\right]dt + [-3x]dw(t)$$

and its solution is as follows (according eq.(3))

$$M_{t,t_o} = e^{\left(\int_0^t\left[\frac{9}{2}-\frac{1}{2}(-3)^2\right]ds + \int_0^t -3dw(s)\right)} = e^{-3w(t)}$$

$$x(t) = e^{-3w(t)}\left[x_o + \int_0^t e^{3w(s)}(-3{x_0}^3+3)ds\right]$$

$$x(t) = e^{-3w(t)}\left[x_o - 3t({x_0}^3-1)e^{3w(t)}\right]$$

$$x(t) = e^{-3w(t)} \quad , \; x_o = \frac{1}{y(0)^3} = 1$$

Since the solution $y(t)$ is the inverse of the transformation $x = U(y) = \frac{1}{y^3}$, then

$$y(t) = \frac{1}{\sqrt[3]{e^{-3w(t)}}}$$

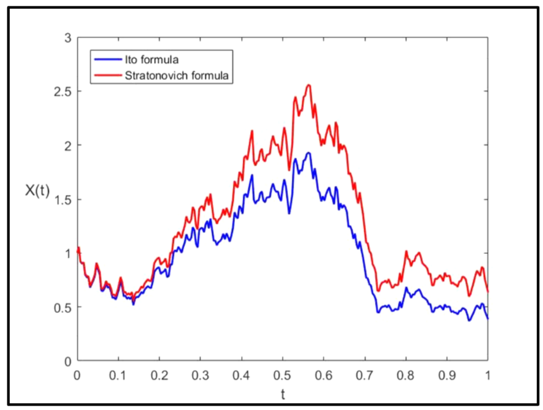

Figure 1.

From the **Figure 1**, we notice that the solutions to both formulas (Ito and Stratonovich) behave the same way during the time interval, and that the difference in the value of $y(t)$ between them increases approximately in the middle of the time interval and decreases at the beginning of this interval.

Example(2): Let we have the following SDEs

$$dy(t) = 3\,e^y dt - 4\,dw \quad , \quad y(0) = 1 \quad \dots\dots\dots(22)$$

And it has solution $y(t)$

Solution:- (Using Ito formula)

$$\hbar(y) = 3\,e^y \quad , \; g(y) = -4$$

We will prove whether the Ito-SDE is reducible or not.

$$\varphi(y) = \frac{\hbar(y)}{g(y)} - \frac{1}{2}g_y(y) = \frac{3\,e^y}{-4} - \frac{1}{2}(0) = -\frac{3}{4}\,e^y$$

$$\Longrightarrow \ \varphi_y(y) = -\frac{3}{4}\, e^y$$

$$\frac{\partial}{\partial y}(g_1) = \frac{\partial}{\partial y}\left(\frac{-\frac{d}{dy}\big(g(y)\varphi_y(y)\big)}{\varphi_y(y)}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{-\frac{d}{dy}(3\, e^y)}{-\frac{3}{4}\, e^y}\right) = 0$$

According to Theorem (1), the Ito-SDE is reducible. We find g_1 from eq.(14)

$$g_1 = \frac{-\frac{d}{dy}\big(g(y)\varphi_y(y)\big)}{\varphi_y(y)} = \frac{-\frac{d}{dy}(3\, e^y)}{-\frac{3}{4}\, e^y} = 4$$

Let $g_2 = 0$, we substitute eq.(10)

)$c = 1$ $\quad (U(y) = c\, e^{g_1 \mathcal{M}(y)} - \frac{g_2}{g_1} = e^{1-y}$

Where

$$\mathcal{M}(y) = \int_{y_0}^{y}\left(\frac{1}{g(s)}\right) ds = \int_{1}^{y}\left(\frac{1}{-4}\right) ds = \frac{1}{-4}(y-1)$$

We apply eq.(13) and we get

$$e^{1-y}\ \left[\left(-\frac{3}{4}\, e^y\right)(4) + \frac{1}{2}(4)^2 - \hbar_1\right] = \hbar_2 - \frac{(\hbar_1)(0)}{(4)}$$

$$-3e + e^{1-y}(8 - \hbar_1) = \hbar_2 \quad \Longrightarrow \quad \hbar_1 = 8 \ \ , \ \ \hbar_2 = -3e$$

Now we substitute $\hbar_1 = 8$, $\hbar_2 = -3e$, $g_1 = 4, g_2 = 0$ in eq.(2)

$$dx(t) = (8\,x - 3e)\ dt + 4x\, dw(t)$$

Thus, the given stochastic equation in Ito form in the question has become linear. Now we find the solution (according eq.(3)) as follows

$$M_{t,t_o} = e^{\left(\int_0^t\left[8 - \frac{1}{2}(4)^2\right]ds + \int_0^t (4) dw(s)\right)} = e^{4w(t)}$$

$$x(t) = e^{4w(t)}\left[1 + \int_0^t e^{-4w(s)}(-3e-4)ds\right]$$

Where $\quad x_o = U(y(0)) = e^{1-y(0)} = 1$, then

$$x(t) = e^{4w(t)} + (-3e-4)t$$

The solution $y(t)$ is the inverse of $x = U(y) = e^{1-y}$, and thus we get the following

$$e^{1-y} = e^{4w(t)} + (-3e-4)t$$

$$y(t) = 1 - \ln\ |e^{4w(t)} + (-3e-4)t|$$

Solution:- (Using Stratonovich formula)

When using Stratonovich's formula, we will solve for the same solution obtained in Ito's formula above (in this example) , because the function $g(y)$ is a constant, and thus we get $\varphi(y) = \rho(y) = \frac{\hbar(y)}{g(y)} = -\frac{3}{4}\ e^y$

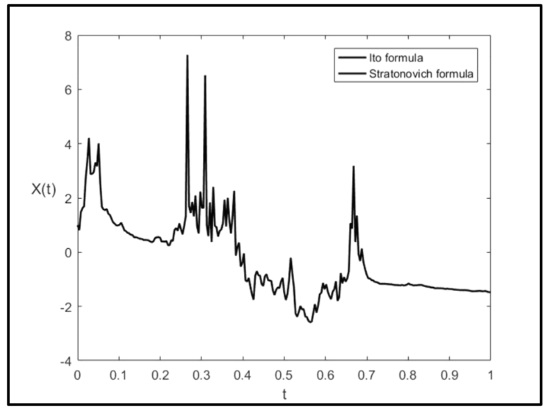

Figure 2.

We notice from the above example that the solutions of the SDE in the two integral forms Ito and Stratonovich are identical because the function $g\big(y(t)\big) = -4$ is a constant function and thus we obtained $\varphi(y) = \rho(y)$, we notice from the **Figure 2** that the solutions behave in an oscillating and decreasing manner until they reach the time value 0.72 and begin to stabilize.

Conclusion

The general formula for solving nonlinear SDEs of the Stratonovich formula was found by the reduction method, and the conditions required for using this method were also determined. When applying the reduction method to both integral formulas (Ito and Stratonovich), we found the following:

1. The reduction method for solving SDEs was generalized from the Ito formula to the Stratonovich formula
2. The reduction method is an effective method for finding solutions to SDEs according to specific conditions explained in Theorem (2) in paragraph 2-3
3. The solution behavior of the stochastic equations of the Stratonovich integral formula does not match the solutions of the Ito formula.
4. The solutions to SDEs are identical when the function $g\big(y(t)\big)$ is constant.

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Conflict of interest

None.

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