On Weakly Regular Rings and SSF-rings

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ABSRACT

In this work we consider weakly regular rings whose simple singular right R-Modules are flat. We also consider the condition (*): R satisfies $L(a)\subseteq r(a)$ for any $a\in R$. We prove that if R satisfies(*) and whose simple singular right R-module are flat, then Z (R) the center of R is a von Neunann regular ring. We also show that a ring R either satisfies (*) or a strongly right bounded ring in which every simple singular right R-module is flat, then R is reduced weakly regular rings.

Keywords: Flat , reduced ,weakly regular rings

حول الحلقات المنتظمة الضعيفة SSF

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الملخص

في هذا البحث درسنا الحلقات المنتظمة الضعيفة, التي تكون مقاساتها الشاذة اليمنى البسيطة مسطحة .وكذلك درسنا الشرط(*)على الحلقةRوالذي يحقق $L(a)\subseteq r(a)$ لكل a في R. وأثبتنا أن الحلقة التي تحقق الشرط (*) والتي تكون مقاساتها الشاذة اليمنى البسيطة مسطحة ,فأن مركز الحلقة يكون حلقة منتظمة. وأثبتنا أيضا إن الحلقة R التي تكون مقاساتها الشاذة اليمنى البسيطة مسطحة و تحقق الشرط (*) أو مقيدة يمنى قوية فأن R حلقة مختزلة ومنتظمة ضعيفة.

الكلمات المفتاحية: المسطحة، المختزلة، حلقات منتظمة ضعيفة.

1- Introduction

Through out this paper, R denotes an associative ring with identity, and all modules are unitary ring R-modules. For any non-empty subset X of ring R, r(X) and L(X) denote the right annihilator of X and the left annihilator of X, respectively. Z(R), J(R) will denote respectively the center of R and Jacobson redical of R. Recall that :- 1- R is called reduced if R has no non-zero nilpotent elements . 2-R is said to be Von Numann regular (or just regular) if , $a\in aRa$ for every $a\in R$, and R is called strongly regular if $a\in a^2R$. Clearly every strongly regular ring is a regular reduced ring. 3- A ring R is said to be right (left) quesi duo ring [3], if every maximal right(left)ideal is a two-sided ideal of R. 4- Following [6], for any ideal of R, R/I is flat if and only if for each $a\in I$,there exists $b\in I$ such that a=ba. 5- A ring R is called weakly right duo[4] if for any $a\in R$, there exists a positive integer n such that a^nR is a two-sided ideal of R.

2- <u>Rings Whose Simple Singular Modules are Flat</u>

<u>Definition 2-1</u>:-

A ring R is called a right (left) SSF-ring ,if every simple singular right (left) R-module is flat.

The following lemma is well-known, so we omit its proof.

<u>Lemma 2-2</u>:-

For any $a\in Z(R)$, if ara=a for some $r\in R$, then there exists $b\in Z(R)$ such that a=aba.

<u>Proof</u>:- see[7]

We consider the condition(*) :R satisfies $L(a) \subseteq r(a)$ for any $a\in R$.

<u>Proposition 2-3</u>:-

If R satisfies (*), SSF-ring, then the center Z(R) of R is a Von Neumann regular ring.

<u>Proof:</u>- First we will show that aR+r(a)=R for any $a\in Z(R)$.

If not, there exists a maximal right ideal M of R such that $aR+r(a) \subseteq M$. Since $a\in Z(R)$, aR+r(a) is an essential right ideal and so M must be an essential right ideal of R. Since R/M is flat and $a\in M$,then there exists $b\in M$ such that a=ba and hence $(1-b) \in L(a) \subseteq r(a) \subseteq M$ implies $1\in M$, which is a contradiction. Therefore aR+r(a)=R for any $a \in Z(R)$and so we have a=ara for some $r\in R$. Applying Lemma (2-2) Z(R) is a Von Neumann regular ring.

Recall that a ring R is right (left) weakly regular if $I^2=I$ for each right (left) ideal I of R; equivalently. $a\in aRaR$ ($a\in RaRa$) for every $a\in R$. R is weakly regular if it is both right and left weakly regular[5].

<u>Lemma 2-4</u>: -

If R satisfies (*), then RaR+r(a) is an essential right ideal.

<u>Proof :-</u> see[7]

<u>Theorem 2-5</u>: -

If R satisfies (*), and SSF-ring, then R is a reduced weakly regular ring.

Proof:- Let $a^2=0$. Suppose that $a\neq 0$.By Lemma (2-4),r(a) is essential right ideal of R. Since $a\neq 0$,r(a) $\neq$R. Thus there exists a maximal essential right ideal M of R containing r(a). Since R/M is flat and $a\in M$ there exists $b\in M$ such that a=ba and hence (1-b) $\in L(a) \subseteq r(a) \subseteq M$ and so $1\in M$, which is a contradiction. Hence a=0 and so R is reduced.

Now, we will show that RaR+r(a)=R for any $a\in R$. Suppose that there exists $b\in R$ such that RbR+r(b) $\neq$R .Then there exists a maximal right ideal M of R containing RbR+r(b). By Lemma(2-4), M must be essential in R. Therefore R/M is flat . Then there exists $c\in M$ such that b=cb and hence(1-c)$\in L(b)\subseteq r(b)\subseteq M$ and so $1\in M$, which is a contradiction. Therefore RaR+r(a)=R for any $a\in R$. Hence R is a right weakly regular ring. Since R is reduced, it also can be easily verified that R is a weakly regular ring.

Corollary 2-6: -

If R is a reduced and SSF-ring, then R is a weakly regular ring.

Lemma 2-7: -

Let R be a right quasi duo ring. Then R/J(R) is a reduced ring.

Proof:- see[6]

Proposition 2-8 :-

Let R be a right quasi duo ring. The following statements are equivalent.

a) R is a right weakly regular ring.

b) R is a strongly regular ring.

Proof :- see[6]

Proposition2-9:-

Every weakly right (left) duo ring is right (left)quasi-duo.

Proof:- see[1]

Proposition 2-10: -

Let R be a right(left) quasi duo ring. If every simple singular right R-module is flat, then R/J(R) is strongly regular.

Proof: - Let $\overline{0}\neq\overline{a}\in\overline{R}=R/J(R)$. We will show that $\overline{R}\overline{a}\overline{R}+r(\overline{a})=\overline{R}$.

Suppose not. Then there exists a maximal right ideal M of R such that $\overline{R}\overline{a}\overline{R}+r(\overline{a})\subseteq M/J(R)$. From Lemma (2-7), $\overline{R}$ is reduced we have $r(\overline{a})=L(\overline{a})$) for any $\overline{a}\in\overline{R}$. Then by Lemma(2-4) $\overline{R}\overline{a}\overline{R}+r(\overline{a})$ is an essential right ideal of $\overline{R}$. Thus R/J(R) must be right essential in $\overline{R}$. Therefore R/M is a simple singular right R-module and so R/M is flat, then there exists $c \in M$ such that a=ca and hence (1-c) $\in L(a) \subseteq r(a) \subseteq M$ and so $I \in M$, which is a contradiction. Therefore R/J(R) is right weakly regular since R/J(R) is reduced it also can be easily verified that R/J(R)is a weakly regular ring. By proposition(2-8), R is a strongly regular ring.

Corollary 2-11: -
Let R be aweakly right duo, SSF-ring. Then R/J(R) is a right weakly regular ring.

Proof:-By Proposition (2-9) R is a right quasi duo ring. Also by Proposition (2-10) R/J(R) is a right weakly regular ring.

A ring R is called strongly right bounded (briefly SRB) [2] if every non-zero right ideal contains a non-zero two-sided ideal of R.

Lemma 2-12:-
If R is a semi prime SRB ring, then R is reduced.

Proof :- see[2]

Theorem 2-13: -

Let R be a SRB and SSF-ring. Then R is a reduced weakly regular.

Proof: - By Corollary(2-6) and Lemma (2-12), it is enough to show that R is semi prime. Suppose that there exists a non-zero right ideal A of R such that $A^2=0$. Then there exists $0 \neq a \in A$ such that $a^2=0$. First observe that r(a) is an essential right ideal of R. If not there exists a non-zero right ideal K such that $r(a) \oplus K$ is right essential in R. Since R is SRB, there is a non-zero ideal I of R such that $I \subseteq K$.

Now $aI \subseteq aR \cap I \subseteq r(a) \cap K = 0$. Hence $I \subseteq r(a) \cap K = 0$ ($aI \subseteq I$).

This is a contradiction. Thus r(a) must be a proper essential right ideal of R. Hence there exists a maximal right ideal M of R containing r(a). Clearly M is an essential right ideal of R, R/M is flat, then there exists $c \in M$ such that a=ca. Now $aca \in aRa \subseteq A^2=0$ and so $1 \in M$, which is a contradiction. Therefore R must be semi-prime, hence R is a reduced weakly regular.

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