On the Rings of Differential Operators

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ABSTRACT

The rings of differential operators have been studied by many mathematicians like Musson [5], Smith and Stafford [7]. Jones in [2] and [3] introduced new ideas for such kind of rings and he found a new line.

In this work, we generalize many of the relations of Jones in the first part, and we found a new proof for some relations of Jones.

Keywords: rings of differential operators.

حول الحلقات من النوع التفاضلي

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الملخص

درسَ الحلقات من النوع التفاضلي المميز الكثيرُ من الباحثين ولعل من أبرزهم مُسن في [5] وسميث وستافورد في[7] بعناية بالغة. وفي [2]و[3] قدم جونس أفكارا أولية لما اعتبر فيما بعد بأنه بداية لخط جديد من هذا النوع من الحلقات. حاولنا في هذا البحث تعميم الكثير من علاقات جونس في الجانب الأول وفي الجانب الثاني تمكنا من ايجاد براهين مختلفة لقسم من هذه العلاقات التي أوجدها جونس في [2] و[3].

الكلمات المفتاحية: الحلقات من النوع التفاضلي.

1. Introduction:

Let k be an algebraically closed field of characteristic zero. For a commutative k-algebra A, we defined $D(A) := \bigcup_{i=0}^{\infty} D^i(A)$ where $D^0(A) = End_A(A)$ and $D^i(A) = \{\theta \in End_k(A) : [\theta, a] \in D^{i-1}(A), \forall a \in A\}$. Then $D(A)$ is a sub ring of $End_k(A)$, called the ring of differential operators on A. For an irreducible affine variety X, we define $D_1(x) :=$ D(O(x)) where O(x) is a ring of regular functions of X and call this ring D(O(x)) of differential operators on X.

Let N be a free Z-module of rank r and $M = Hom_z(N, Z)$ its Z-module dual. Then we have a bilinear pairing $\langle , \rangle : M \times N \to Z$ which extends to $\langle , \rangle : M_Q \times N_Q \to Q$ where $M_Q = M \otimes_Z Q$ and $N_Q = N \otimes_Z Q$. Let $f \in (M_Q)^*$ be a subset of the form $H_f = \{\lambda \in M_Q : f(\lambda) \geq 0\}$, defined is a half space of M_Q. Let V be a finite dimensional vector space over Q.

A subset $\{v \in V : \lambda(v) \geq 0\}$ where $\lambda : V \to Q$ is a non-zero linear function called a half space of V. A cone σ, in V is the intersection of a finite number of half spaces. It can be written in the form:

$$\sigma = \sum_{i=1}^{s} Q_{\geq 0}\, v_i \quad \text{for } v_1, v_2, \ldots\ldots, v_s \in V.$$

A subset of σ of the form $\{v \in V : \lambda(v) = 0\}$ where $\lambda : V \to Q$ is a linear functional positive on σ is called a face of σ. The dimension of the cone σ is the dimension of the vector space σ-σ over Q.

Consider a cone σ in N_Q. Let $\check{\sigma} = \{p \in M_Q : \langle p, \sigma \rangle \subseteq Q_{\geq 0}\}$. Then $\check{\sigma}$ is an r-dimensional cone in M_Q and $\check{\sigma} \cap M$ is finitely generated additive semi-group containing 0, (see [6, 1.1]). Let $R = k[\check{\sigma} \cap M] = \oplus_{\lambda \in \check{\sigma} \cap M} X$, be the semi-group algebra.

Here X^{λ} is a formal monomial and the multiplication is given by the semi-group addition. Choose once and for all a Z-basis of M, say $\{m_1, m_2, \ldots\ldots\ldots m_r\}$. Then set $x_i = X^{m_i}$ and $\partial_i = \partial / \partial x_i$.

Jones in [3] starts with a single relation. In this paper, we use that relation to obtain a new general basis.

2. The semi-groups Λ and $\tilde{\Lambda}$

In this section, we repeat in the first part the definition of the semi-group Λ, we give many of its features. Jones in [3] used these features to define another semi-group, denoted by $\tilde{\Lambda}$.

Consider a finite set F_1 of hyper-planes such that each is parallel but not equal to some ∂H_i. We also suppose that $\Lambda := \left(\check{\sigma} \cap M\right) \setminus \left(\bigcup_{F \in F_1} F \cap M\right)$ is a semi-group. We call such a semi-group a hyper-plane deleted sub semi-group of $\left(\check{\sigma} \cap M\right)$. For any hyper-plane F in M_Q, let e(F) be the polynomial of $S(M_Q^*)$ which defines F. Note that for $F \in F_1$, $e(F)$ are the polynomials of degree 1, with rational coefficients.

Definition 2.1: A semi-group Γ is said to be normal if one of the following equivalents conditions holds:

(1) For $a, b, c \in \Gamma$ if $a + nb = nc$ for $n \in N$, then $a = na_1$ for some $a_1 \in \Gamma$.

(2) For $a \in Z\Gamma$ and $0 \neq n \in N$, if $na \in \Gamma$, then $a \in \Gamma$.

For an arbitrary finitely generated semi-group the normalization of $\Gamma, \tilde{\Gamma}$ is defined as: $\tilde{\Gamma} = \{a \in Z\Gamma : na \in \Gamma, \text{ for some } 0 \neq n \in N\}$.

Musson in [4,1.3] introduced the following proposition:

Proposition 2.2: For a semi-group Λ, the following are equivalents:

(1) Λ is normal;
(2) For any field k, $k\Lambda$ is an integrally closed Noetherian domain;
(3) For some $0 \leq t \leq s$, Λ is isomorphic to a semi-group of the form $\left(Z_{\geq 0}^{t} \times Z^{s-t}\right) \cap V$, where V is a subspace of M_Q and dim $Q\Lambda = r$;
(4) For some $t, n \geq 0$, Λ is isomorphic to a semi-group of the form $M \cap H_1 \cap \ldots \cap H_t$ where H_i are half spaces in M_Q and $Z\Lambda = M$.

In the following lemma, Jones in [3] defined another semi-group $\tilde{\Lambda}$:-

Lemma 2.3:

(1) $\tilde{\Lambda} := \check{\sigma} \cap M$;

(2) $k\Lambda = k\left[\check{\sigma} \cap M\right] = k\tilde{\Lambda}$.

Proof:

(1) For all $\lambda \in \check{\sigma} \cap M$ and $a \in N$, we have $a\lambda \in \Lambda$ because Λ is a semi-group of $\check{\sigma} \cap M$ such that $\otimes_{\geq 0} \Lambda = \check{\sigma}$ and $Z\Lambda = M$, then $M = Z\Lambda = Z\tilde{\Lambda}$ and $M = (\check{\sigma} \cap M)$, therefore $\check{\sigma} \cap M = \tilde{\Lambda}$.

(2) The second result is true by the proposition (2.2) in the second part ■

Now, For Λ an arbitrary semi-group such that $Z\Lambda = M$, let $k\Lambda$ be the associated semi-group algebra of Λ and $D(k\Lambda)$ the ring of differential operators. Then $D(k\Lambda) \subseteq D(kM)$

$$= D(k[x_1^{\pm 1},........, x_r^{\pm 1}])$$
$$= k[x_1^{\pm 1},......, x_r^{\pm 1}, \partial_1,......., \partial_r].$$

Clearly $x_i\partial_i$ is in $D(k\Lambda)$. Now for $\mu \in Z\Lambda$, $x_i\partial_i * x^{\mu} = \mu_i x^{\mu}$, where $\mu = (\mu_1, \mu_2, ..., \mu_r)$.

Let $W = Q[x_1\partial_1,.......x_r\partial_r] \otimes_Q k$ then $W \subseteq D(k\Lambda)$. Thus the elements of W define polynomial functions from $Z\Lambda \subseteq M_Q$ to k by the rule $x_i\partial_i(\mu) = \mu_i$ for $\mu \in Z\Lambda$.

Thus for $f \in W$ and $\mu \in Z\Lambda$, $f(x_1\partial_1,...., x_r\partial_r)x^{\mu} = f(\mu)x^{\mu}$.

Definition 2.4: For $A, B \subseteq M$ and $\lambda \in M$, define

$\Omega_{A,B}(\lambda) = \{\mu \in A : \lambda + \mu \notin B\}$

Also let

$\Omega_A(\lambda) = \Omega_{A,A}(\lambda)$. For $\Omega \subseteq M$ define $\overline{\Omega}$ to be the Zariski closure in M_Q and let $I(\Omega) = \{f \in W : f(\rho) = 0, \forall \rho \in \Omega\}$. Also $I(\Omega) = I(\overline{\Omega})$.

Lemma 2.5: [Musson]

Let Λ be a semi-group of M with $Q\Lambda = M_Q$. For $g \in (M_Q)^*$ and $b \in Q$ set $\Lambda_b = \{\lambda \in \Lambda : g(\lambda) = b\}$. Suppose that:

(1) $\Lambda_b \neq \phi$;

(2) $\dim_Q Q\Lambda_0 = r - 1$. Then $\overline{\Lambda_b} = \{\lambda \in M_Q : g(\lambda) = b\}$

Now, we decompose $\Omega_{\tilde{\Lambda}}(\lambda)$ into pieces,

$\Omega^i_{\tilde{\Lambda}}(\lambda) = \{\mu \in \tilde{\Lambda} : \lambda + \mu \notin H_i \cap M\}$.

Observer that $\Omega_{\tilde{\Lambda}}(\lambda) = \cup \Omega^i_{\tilde{\Lambda}}(\lambda)$.

Proposition2.6:
$\overline{\Omega^{i}_{\tilde{\Lambda}}(\lambda)} = \{\mu \in M_Q : h_i(\mu) \in h_i(M) \ \&\ 0 \le h_i(\mu) < -h_i(\lambda)\}$, for i=1,2,…,r.
Hence $\overline{\Omega^{i}_{\tilde{\Lambda}}(\lambda)}$ is a finite union of hyper-planes parallel to ∂H_i.

Proof: h_i is linear, then the left side is included in the right-hand side. Also, the right-hand side is Zariski closure in M_Q. By the Lemma 2.5,we suppose that $h_i(\mu) = b \in h_i(M)$ with $0 \le h_i(\mu) < -h_i(\lambda)$. Let $\Lambda_b = \{\lambda \in \Lambda : h_i(\mu) = b\}$ with $\Lambda_b \ne \phi$. ∂H_i is a face of σ, then $\dim_Q Q\Lambda_0 = r-1$. Therefore, by the Lemma 2.5, $\overline{\Lambda_b} = \{\lambda \in M_Q : h_i(\lambda) = b\}$ and $\mu \in \Omega^{i}_{\tilde{\Lambda}}(\lambda)$.

Proposition 2.7:

(1) $W \subseteq D_1(x)$;

(2) $D_1(x) = \bigoplus_{\lambda \in M} X^{\lambda}\ I(\overline{\Omega_{\tilde{\Lambda}}(\lambda)})$;

(3) $\overline{\Omega_{\tilde{\Lambda}}(\lambda)}$ is a finite union of hyper-planes each parallel to some ∂H_i;

(4) $\Omega_{\tilde{\Lambda}}(\lambda) = \overline{\Omega_{\tilde{\Lambda}}(\lambda)} \cap \tilde{\Lambda}$.

Proof: Clearly the first and the second properties are satisfied by [2]. For the third property, we have $\Omega(\lambda) = \cup \Omega^{i}(\lambda)$. By the proposition 2.6, $\overline{\Omega^{i}_{\tilde{\Lambda}}(\lambda)}$ is a finite union of hyper-planes parallel to ∂H_i, then

$$\overline{\Omega(\lambda)} = \overline{\cup \Omega^{i}(\lambda)} = \cup \overline{\Omega^{i}(\lambda)},$$

and $\overline{\Omega(\lambda)}$ is a finite union of hyper-planes parallel to ∂H_i. For the fourth property, it's clear that $\Omega_{\tilde{\Lambda}}(\lambda) \subseteq \overline{\Omega_{\tilde{\Lambda}}(\lambda)} \cap \tilde{\Lambda}$.

Now we suppose that $F \subseteq \overline{\Omega_{\tilde{\Lambda}}(\lambda)}$ is a finite union of hyper-planes parallel to some ∂H_i. By [3, lemma 3.5] we have

$$\lambda + F \subseteq H_i^c \Rightarrow \lambda + (F \cap \tilde{\Lambda}) \subseteq H_i^c$$

$$\Rightarrow F \cap \tilde{\Lambda} \subseteq \Omega(\lambda).$$

and $\overline{\Omega_{\tilde{\Lambda}}} \cap \tilde{\Lambda} = (\cup F) \cap \tilde{\Lambda}$

$= \cup (F \cap \tilde{\Lambda})$

$\subseteq \Omega_{\tilde{\Lambda}}(\lambda)$.

Then $\Omega_{\tilde{\Lambda}}(\lambda) = \overline{\Omega_{\tilde{\Lambda}}(\lambda)} \cap \tilde{\Lambda}$.

3. <u>The new relations</u>:

In this section, we start with a single relation of Jones and we use this relation to obtain a new general basis.

<u>Proposition 3.1</u>: For $\lambda \in M = Z\Lambda = Z\tilde{\Lambda}$, the following holds:

(1) $\overline{\Omega_{\tilde{\Lambda}}(\lambda)} = \overline{\Omega_{\Lambda,\tilde{\Lambda}}(\lambda)} \cap \left(\bigcup_{F \in Y} F \right)$ where $Y = \left\{ F \in F_1 : F \subseteq \overline{\Omega_{\tilde{\Lambda}}(\lambda)} \right\}$.

(2) $\overline{\Omega_{\tilde{\Lambda},\Lambda}(\lambda)} = \overline{\Omega_{\tilde{\Lambda}}(\lambda)} \cup \bigcup_{F \in \Sigma} (F - \lambda)$ where $\Sigma = \left\{ F \in F_1 : (F - \lambda) \cap \tilde{\Lambda} \neq \phi \right\}$.

<u>Proof:</u>

(1) Let Φ be the set of hyper-plane, then

$$\overline{\Omega_{\tilde{\Lambda}}(\lambda)} = \bigcup_{F \in \Phi} F = \bigcup_{F \in Y} F \cup \bigcup_{F \in \Phi \backslash Y} F.$$

Now, by the definition $\Omega_{\Lambda,\tilde{\Lambda}}(\lambda) = \left\{ \mu \in \Lambda : \lambda + \mu \notin \tilde{\Lambda} \right\}$, then $\Omega_{\Lambda,\tilde{\Lambda}}(\lambda) = \Omega_{\tilde{\Lambda}}(\lambda) \cap \Lambda$.

Also $\overline{\Omega_{\Lambda,\tilde{\Lambda}}(\lambda)} = \overline{\Omega_{\tilde{\Lambda}}(\lambda) \cap \Lambda}$

$= \overline{\overline{\Omega_{\tilde{\Lambda}}(\lambda)} \cap \tilde{\Lambda} \cap \Lambda}$;by (2.7.4)

$= \overline{\overline{\Omega_{\tilde{\Lambda}}(\lambda)} \cap \Lambda}$; $(\Lambda \subseteq \tilde{\Lambda})$

$= \overline{(\bigcup_{F \in \Phi \backslash Y} F \cup \bigcup_{F \in Y}) \cap \Lambda}$

$= \overline{(\bigcup_{F \in \Phi \backslash Y} F \cap \Lambda) \cup (\bigcup_{F \in \Phi \backslash Y} F \cap \Lambda)}$

$= \overline{\bigcup_{F \in \phi \backslash Y} F \cap \Lambda}$

$= \bigcup_{F \in \phi \backslash Y} F$

Thus $\overline{\Omega_{\tilde{\Lambda}}(\lambda)} = \overline{\Omega_{\Lambda,\tilde{\Lambda}}(\lambda)} \cap \bigcup_{F \in Y} F$.

(2) See [3, Proposition 3.11.2] ■

Jones in [3] gave the following lemma, which is given here with a new proof.

Lemma 3.2: For $\lambda \in M, \overline{\Omega_{\Lambda}(\lambda)} = \overline{\Omega_{\Lambda,\tilde{\Lambda}}(\lambda)} \cup \bigcup_{F\in\Sigma'}(F-\lambda)$ where $\Sigma' = \{F \in F_1 : (F-\lambda) \cap \Lambda \neq \phi\}$

Proof: We have, $\Omega_{\tilde{\Lambda}}(\lambda) = \Omega_{\tilde{\Lambda},\Lambda}(\lambda) \cap \Lambda$. By proposition (3.1):

$$\Omega_{\Lambda}(\lambda) = \left(\Omega_{\tilde{\Lambda}}(\lambda) \cup \bigcup_{F\in\Sigma}(F-\lambda)\right) \cap \Lambda$$

$$= \left(\Omega_{\tilde{\Lambda}}(\lambda) \cap \Lambda\right) \cup \left(\bigcup_{F\in\Sigma'}(F-\lambda)\right)$$

$$= \Omega_{\Lambda,\tilde{\Lambda}}(\lambda) \cup \bigcup_{F\in\Sigma'}(F-\lambda)$$

From Proposition (2.7) and (3.1) and Lemma 3.2, we can write and prove the following:

Theorem 3.3:

(1) $\Omega_{\Lambda}(\lambda) \cap \Omega_{\tilde{\Lambda}}(\lambda) = \Omega_{\Lambda,\tilde{\Lambda}}(\lambda)$;

(2) $\Omega_{\Lambda}(\lambda) \cap \tilde{\Lambda} = \Omega_{\Lambda}(\lambda)$;

(3) $\Omega_{\tilde{\Lambda},\Lambda}(\lambda) \cap \tilde{\Lambda} = \Omega_{\tilde{\Lambda},\Lambda}(\lambda)$;

(4) $\Omega_{\Lambda,\tilde{\Lambda}}(\lambda) \cap \tilde{\Lambda} = \Omega_{\Lambda,\tilde{\Lambda}}(\lambda)$;

(5) $\Omega_{\Lambda}(\lambda) \cap (\tilde{\Lambda} \setminus \Lambda) = \phi$;

(6) $\Omega_{\Lambda,\tilde{\Lambda}}(\lambda) \cap (\tilde{\Lambda} \setminus \Lambda) = \phi$;

(7) $\Omega_{\tilde{\Lambda},\Lambda}(\lambda) \cap (\tilde{\Lambda} \setminus \Lambda) = \Omega_{\tilde{\Lambda},\Lambda}(\lambda) \setminus \Omega_{\Lambda}(\lambda)$.

Proof:

(1) Since $\Omega_{\Lambda}(\lambda) = \Omega_{\tilde{\Lambda},\Lambda}(\lambda) \cap \Lambda$ *and* $\Omega_{\Lambda,\tilde{\Lambda}}(\lambda) = \Omega_{\tilde{\Lambda}}(\lambda) \cap \Lambda$, then by Proposition 3.1 and Lemma 3.2, we have

$\Omega_{\Lambda,\tilde{\Lambda}}(\lambda) \subseteq \Omega_{\Lambda}(\lambda) \subseteq \Omega_{\tilde{\Lambda},\Lambda}(\lambda)$

$\Omega_{\Lambda,\tilde{\Lambda}}(\lambda) \subseteq \Omega_{\tilde{\Lambda}}(\lambda) \subseteq \Omega_{\tilde{\Lambda},\Lambda}(\lambda)$

Then $\Omega_{\Lambda}(\lambda) \cap \Omega_{\tilde{\Lambda}}(\lambda) = (\Omega_{\tilde{\Lambda},\Lambda}(\lambda) \cap \Lambda) \cap \Omega_{\tilde{\Lambda}}(\lambda)$

$$= \Omega_{\tilde{\Lambda},\Lambda}(\lambda) \cap (\Omega_{\tilde{\Lambda}}(\lambda) \cap \Lambda)$$
$$= \Omega_{\tilde{\Lambda},\Lambda}(\lambda) \cap \Omega_{\Lambda,\tilde{\Lambda}}(\lambda)$$
$$= \Omega_{\Lambda,\tilde{\Lambda}}(\lambda).$$

(2) $\Omega_{\Lambda}(\lambda) \cap \tilde{\Lambda} = \Omega_{\tilde{\Lambda},\Lambda}(\lambda) \cap \Lambda \cap \tilde{\Lambda}$
$$= \Omega_{\tilde{\Lambda},\Lambda}(\lambda) \cap \Lambda = \Omega_{\Lambda}(\lambda).$$

(3) $\Omega_{\tilde{\Lambda},\Lambda}(\lambda) \cap \tilde{\Lambda} = \left(\left(\bigcup_{F \in F_1}(F-\lambda) \cap \tilde{\Lambda}\right) \cup \Omega_{\tilde{\Lambda}}(\lambda)\right) \cap \tilde{\Lambda}$
$$= (\bigcup_{F \in F_1}(F-\lambda) \cap \tilde{\Lambda}) \cup (\Omega_{\tilde{\Lambda}}(\lambda) \cap \tilde{\Lambda})$$
$$= (\bigcup_{F \in F_1}(F-\lambda) \cap \tilde{\Lambda}) \cup \Omega_{\tilde{\Lambda}}(\lambda)$$
$$= \Omega_{\tilde{\Lambda},\Lambda}(\lambda).$$

(4) $\Omega_{\Lambda,\tilde{\Lambda}}(\lambda) \cap \tilde{\Lambda} = (\Omega_{\Lambda}(\lambda) \cap \Omega_{\tilde{\Lambda}}(\lambda)) \cap \tilde{\Lambda}$, by (1)
$$= (\Omega_{\Lambda}(\lambda) \cap \tilde{\Lambda}) \cap \Omega_{\tilde{\Lambda}}(\lambda)$$
$$= \Omega_{\Lambda}(\lambda) \cap \Omega_{\tilde{\Lambda}}(\lambda) \quad , \quad \text{by(2)}$$
$$= \Omega_{\Lambda,\tilde{\Lambda}}(\lambda).$$

(5) $\Omega_{\Lambda}(\lambda) \cap (\tilde{\Lambda} \setminus \Lambda) = (\Omega_{\Lambda}(\lambda) \cap \tilde{\Lambda}) - (\Omega_{\Lambda}(\lambda) \cap \Lambda)$
$$= \Omega_{\Lambda}(\lambda) - \Omega_{\Lambda}(\lambda) \quad , \quad \text{by(2)}$$
$$= \phi.$$

(6) $\Omega_{\Lambda,\tilde{\Lambda}}(\lambda) \cap (\tilde{\Lambda} \setminus \Lambda) = (\Omega_{\Lambda,\tilde{\Lambda}}(\lambda) \cap \tilde{\Lambda}) - (\Omega_{\Lambda,\tilde{\Lambda}}(\lambda) \cap \Lambda)$
$$= \phi. \quad , \quad \text{by(4)}$$

(7) $\Omega_{\tilde{\Lambda},\Lambda}(\lambda) \cap (\tilde{\Lambda} \setminus \Lambda) = (\Omega_{\tilde{\Lambda},\Lambda}(\lambda) \cap \tilde{\Lambda}) - (\Omega_{\tilde{\Lambda},\Lambda}(\lambda) \cap \Lambda)$
$$= \Omega_{\tilde{\Lambda},\Lambda}(\lambda) - \Omega_{\Lambda}(\lambda), \text{ by (3)}$$
$$= \Omega_{\tilde{\Lambda},\Lambda}(\lambda) \setminus \Omega_{\Lambda}(\lambda) \ .$$

We conclude the following theorem:

Theorem 3.4:

(1) $\Omega_{\Lambda}(\lambda)\cap(\bigcup_{F\in\Sigma}(F-\lambda))=\bigcup_{F\in\Sigma'}(F-\lambda)$.

(2) $\Omega_{\Lambda,\tilde{\Lambda}}(\lambda)\cap(\bigcup_{F\in\Sigma}(F-\lambda))=\Omega_{\tilde{\Lambda}}(\lambda)\cap(\bigcup_{F\in\Sigma'}(F-\lambda))$

Proof:

(1)
$$\begin{aligned}\Omega_{\Lambda}(\lambda)\cap(\bigcup_{F\in\Sigma}(F-\lambda)) &= (\Omega_{\Lambda,\tilde{\Lambda}}(\lambda)\cup\bigcup_{F\in\Sigma'}(F-\lambda))\cap(\bigcup_{F\in\Sigma}(F-\lambda))\\ &=(\Omega_{\Lambda,\tilde{\Lambda}}(\lambda)\cap(\bigcup_{F\in\Sigma}(F-\lambda)))\cup(\bigcup_{F\in\Sigma'}(F-\lambda)\cap\bigcup_{F\in\Sigma}(F-\lambda)))\\ &=(\Omega_{\Lambda,\tilde{\Lambda}}(\lambda)\cap(\bigcup_{F\in\Sigma}(F-\lambda)))\cup(\bigcup_{F\in\Sigma'}(F-\lambda))\\ &=(\Omega_{\tilde{\Lambda}}(\lambda)\cap\Lambda\cap(\bigcup_{F\in\Sigma}(F-\lambda)))\cup(\bigcup_{F\in\Sigma'}(F-\lambda))\\ &=(\Omega_{\tilde{\Lambda}}(\lambda)\cap(\bigcup_{F\in\Sigma'}(F-\lambda)))\cup(\bigcup_{F\in\Sigma'}(F-\lambda))\\ &=\bigcup_{F\in\Sigma'}(F-\lambda).\end{aligned}$$

(2)
$$\begin{aligned}\Omega_{\Lambda,\tilde{\Lambda}}(\lambda)\cap(\bigcup_{F\in\Sigma}(F-\lambda)) &= \Omega_{\Lambda,\tilde{\Lambda}}(\lambda)\cap(\bigcup_{F\in F_1}(F-\lambda)\cap\tilde{\Lambda})\\ &=\Omega_{\Lambda,\tilde{\Lambda}}(\lambda)\cap(\bigcup_{F\in F_1}(F-\lambda)), \qquad \text{by}(3.3.4)\\ &=(\Omega_{\tilde{\Lambda}}(\lambda)\cap\Lambda)\cap(\bigcup_{F\in F_1}(F-\lambda))\\ &=\Omega_{\tilde{\Lambda}}(\lambda)\cap(\bigcup_{F\in\Sigma'}(F-\lambda))\end{aligned}$$

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