On Annihilating - Ideal Graph of Z_n

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ABSTRACT

In this paper, we study and give some properties of annihilating-ideal graphs of Z_n, also we find Hosoya polynomial and Wiener index for this graph.

Key word: Annihilating – ideal graph, Z_n, Hosoya polynomial, Wiener index.

بيان تالف المثاليات لحلقات Z_n

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الملخص

في هذا البحث، ندرس ونعطي بعض الخواص لبيانات تالف المثاليات لحلقات Z_n، كذلك نجد الهوسويا ودليل وينر لهذه البيانات.

الكلمات المفتاحية: بيان تلف المثاليات، Z_n , متعددة حدود هوسويا ، دليل وينر.

1. Introduction

In this paper (α) be the ideal of R generated by α and" $A^*(R)$ be the set of non-zero ideals with non-zero annihilators. We associate a simple graph AG(R) with vertices $A^*(R)$ and two ideal vertices I_1 and I_2 are adjacent if and only if $I_1I_2= (0)$ [2]".

Recall according to [3]

1- Let G (V, E) be a simple graph with vertices set V and edges set E be **connected** if there is a path between any two distinct vertices of G. For a vertices x and y of G, denoted d(x,y) be the length of a shortest path form x to y.

2- The **diameter** of G denoted by diam (G) and equal max{d(x,y): x and y are vertices of G}.

3- The **degree** of $x\in$ V (G) is denoted by deg(x) and it is the number of vertices who's adjacent with x in G.

4- If G_1 and G_2 are graphs, then we say that **G_1 is an isomorphic to G_2** , (or $G_1\cong G_2$) ,if there exists a one-to -one mapping φ from $V(G_1)$ onto $V(G_2)$ such that φ preserves the adjacent.

5- The complete sub-graph K_t of any graph G is called a **clique**, **and ω(G) is the clique number of G** , which is the greatest integer $t \geq 1$, such that $K_t \subseteq G$.

As usual, we shall assume that p and q are distinct prime numbers. $\lceil s\rceil$($\lfloor s\rfloor$ resp.) It means that the smallest integer is not less than s (the greatest integer is not greater than s resp.). Z_n we denote a ring of integer modulo n, . By [4] any ideal of Z_n is principal and Z_n local ring if and only if $n=p^m$, where m is a positive integer number. In [2] Behboodi and Rakeei show that for every ring R, the annihilating-ideal graph AG(R) is connected and diam(AG(R))$\leq$3.

2. on Annihilating-Ideal Graph of Z_{p^m}.

In this section we give some basic properties of annihilating ideal graph of Z_{p^m}. First we give an order and size of AG (Z_{p^m}), where m≥4.

Theorem 2.1:

If R=Z_{p^m}, then AG(R) has order m-1 vertices and the size a_1, where

$$a_1=\begin{cases}\frac{m(m-2)}{4}, & \text{if m even,}\\ \frac{(m-1)^2}{4}, & \text{if m odd.}\end{cases}$$

Proof: Since any ideal of Z_n is principal , then clearly that the ideals of ring Z_{p^m} are{(p),(p^2),…,(p^{m-1})}, therefor AG(R) has (m-1) vertices ideal.

Now to find size of graph AG(R).

Since $p^m \equiv 0 \bmod p^m$, then (p^i)(p^j)=0 iff i+j ≥ m, where 1≤i,j≤ m-1, so that (p^i) adjacent with (p^j) whenever 1≤i< m-1 and j= (m-i),…,m-1, which implies that $\sum_{j=m-i}^{m-1} 1 = (m-1)-(m-i)+1 = i$, and since ($p^i$) ($p^i$) =0 if and only if i≥ $\left\lceil\frac{m}{2}\right\rceil$.

Then we have (p^i) has loop, so that

$$\deg\big((p^i)\big)=\begin{cases} i\ , & \text{if } 1\le i\le \left\lceil\frac{m}{2}\right\rceil-1\ ,\\ i-1, & \text{if } \left\lceil\frac{m}{2}\right\rceil \le i\le m-1\ ,\end{cases}$$

and

$$\sum_{v\in AG(z_{p^m})}\deg v= \sum_{i=1}^{\left\lceil\frac{m}{2}\right\rceil-1} i+\sum_{i=\left\lceil\frac{m}{2}\right\rceil}^{m-1}(i-1)=\sum_{i=1}^{m-1} i-\sum_{i=\left\lceil\frac{m}{2}\right\rceil}^{m-1}1=\frac{m(m-3)}{2}+\left\lceil\frac{m}{2}\right\rceil$$

Since $\sum_{v\in AG(z_{p^m})}\deg v=2a_1$, where a_1 is the number of edges of AG(Z_{p^m}), then $a_1=\frac{m(m-2)}{4}$, when m is even, and $a_1=\frac{(m-1)^2}{4}$, when m is odd. ■

The next result we give the cliqu**e** number of AG(Z_{p^m}).

Theorem 2.2

For any positive number m≥4, AG(Z_{p^m}) contains a sub-graph $K_{\left\lceil\frac{m}{2}\right\rceil}$ and ω(AG(Z_{p^m}))=$\left\lceil\frac{m}{2}\right\rceil$

Proof : There are $p^{\left\lceil\frac{m}{2}\right\rceil}$elements divisible by $p^{\left\lceil\frac{m}{2}\right\rceil}$, we can write $V=\{v_1,v_2,\dots,v_{\left\lceil\frac{m}{2}\right\rceil}\}$, where v_i= (p^{m-i}) , 1 ≤ i ≤ $\left\lceil\frac{m}{2}\right\rceil$ this means that all ideal vertices of V are adjacent each other .Thus V = V($k_{\left\lceil\frac{m}{2}\right\rceil}$) a complete sub graph of AG(Z_{p^m}). Also any ideal vertices x=(p^{m-i}), $\left\lceil\frac{m}{2}\right\rceil$+1≤i≤ m-1 are non–adjacent with ideal vertex $v_{\left\lceil\frac{m}{2}\right\rceil}=(p^{\left\lceil\frac{m}{2}\right\rceil})$, so that , $\left\lceil\frac{m}{2}\right\rceil$ greats integer such that $k_{\left\lceil\frac{m}{2}\right\rceil}$ ⊆AG(Z_{p^m}), whence ω(AG(Z_{p^m}))=$\left\lceil\frac{m}{2}\right\rceil$ ■

Recall that "**radius** of G is rad (G) = min{d(x,y) : x and y are vertices of G} and **the center** of G is defined by Cent(G) = {x∈ V(G) : d(x,y)= rad (G) for any y∈V(G)}[3]".

Proposition2.3

For any positive number m≥4, rad(AG(Z_{p^m}))=1 and Cent(AG(Z_{p^m}))=(p^{m-1}).

Proof: Since Z_{p^m} be a local ring, then by [7] every minimal ideal vertex adjacent with every ideal vertices in $AG(Z_{p^m})$. That means that the graph $AG(Z_{p^m})$ contains subgraph $K_{1,(m-2)}$, so rad $(AG(z_{p^m}))=1$ and hence $(p^{m-1})\subseteq Cent(AG(Z_{p^m}))$ …(1).

Now, let $x\in AG(Z_{p^m})$ such that $x \notin (p^{m-1})$ and for any $y\in (p)$, then $xy\neq 0$, this means that $x \notin Cent(AG(Z_{p^m}))$ and hence $Cent(AG(Z_{p^m}))\subseteq (p^{m-1})$…(2).
Form (1) and (2), we get $Cent(AG(Z_{p^m})) = (p^{m-1})$ ■

Proposition 2.4
For any positive number m≥4, we have $diam(AG(Z_{p^m}))=2$
Proof: Clearly the ideal vertex (p^{m-1}) adjacent with every ideal vertices in $AG(Z_{p^m})$, also the ideal vertices $\alpha_1=(p)$ and $\alpha_2=(p^2)$ are non-adjacent, then we have $diam(AG(Z_{p^m}))=2$.

Theorem 2.5
For any positive number m≥4, $AG(Z_{p^m})\cong (AG(Z_{p^{m-1}}) \cup \{K_1,U_s\})$,where $s = \left\lceil\frac{m}{2}\right\rceil-1$ and U_s is added edges are numbered s that connecting K_1 to (p^{m-i}) where $i =1,2,\ldots,\left\lceil\frac{m}{2}\right\rceil$.
Proof: We observe that the number of vertices of two graph are equal (m-1). In $AG(Z_{p^m})$ the ideal vertices (p^i) and (p^j) are adjacent if and only if $i+j \geq m$, also in $G=(AG(Z_{p^{m-1}}) \cup\{K_1,U_s\})$, the ideal vertices $u_i= (p^i)$ and $u_j= (p^j)$ are adjacent if and only if $i+j \geq m-1$, or $u_i= w$, where $V(K_1)=w$, $i = \left\lceil\frac{m}{2}\right\rceil$and $j = \left\lceil\frac{m}{2}\right\rceil,\ldots,m-2$. Therefor we can label the vertices in $AG(Z_{p^m})$ are $v_i= (p^i)$, i=1, …, m-1 and the vertices in G are

$$u_i = \begin{cases} (p^i), & \text{if } i = 1,2,\ldots,\left\lceil\frac{m}{2}\right\rceil - 1, \\ w, & \text{if } i = \left\lceil\frac{m}{2}\right\rceil, \\ (p^{i-1}), & \text{if } i = \left\lceil\frac{m}{2}\right\rceil + 1,\ldots,m-1. \end{cases}$$

We can defined a mapping f from $AG(Z_{p^m})$ to, $G= (AG(Z_{p^{m-1}}) \cup \{K_1,U_s\})$, such that $f(v_i)=u_i$, where $1\leq i \leq m-1$.

Clearly that f is onto and one-to-one, we only prove that if e=uv is an edge in $AG(Z_{p^m})$ then f(e)=f(u)f(v) is an edge in G.

Let $e =v_iv_j$ edge from v_i to v_j in $AG(Z_{p^m})$, therefore $i+j\geq m$ we get two cases.

<u>Case 1:</u> If i or j = $\left\lceil\frac{m}{2}\right\rceil$, say i = $\left\lceil\frac{m}{2}\right\rceil$ then v_j has two sub cases a and b:
<u>Sub-case a:</u> If m is an even number then $\left\lceil\frac{m}{2}\right\rceil = \frac{m}{2}$, so v_i adjacent with v_j, where $j=(\frac{m}{2}+1)$,..., m-1, on the other hand $i= \frac{m}{2}$, $f(v_i) = w$ and $f(v_j)= u_j =(p^{j-1})$for all $j=(\frac{m}{2} + 1)$,…,m-1. From defined G, we see w adjacent with $u_j= (p^{j-1})$, $j=\frac{m}{2} + 1,\ldots,m-1$, therefor if e is an edge in $AG(Z_{p^m})$, then f(e) is an edge in G.
<u>Sub-case b:</u> If m is odd then $\left\lceil\frac{m}{2}\right\rceil = \frac{m+1}{2}$, the vertex v_i adjacent with v_j if and only if $j = \frac{m-1}{2},\frac{m+1}{2},\frac{m+1}{2} + 1,\ldots,m-1$, except case, $\frac{(m+1)}{2}$ because i=j , v_i adjacent with v_j . Where $j = \frac{(m-1)}{2},(\frac{m+1}{2} + 1),\ldots,m-1$. On the other hand since $i =(\frac{m+1}{2})$ then $f(v_i)=w$ and

$$f(v_j)=\begin{cases}(p^j), & \text{if} \quad j=\frac{m-1}{2},\\ (p^{j-1}), & \text{if} \quad j=\frac{m+1}{2}+1,\ldots,m-1.\end{cases}$$

From defined G, observe that w adjacent with u_j, $j=\frac{(m-1)}{2}, \frac{(m+1)}{2}+1,\ldots, m-2$. That mean if e is an edge in $AG(Z_{p^m})$ then f (e) is an edge in G.

<u>Case 2</u>: If i and $j \neq \left\lceil\frac{m}{2}\right\rceil$, then e edge in $AG(Z_{p^m})$ iff $i+j \geq m$ and $i,j \neq \left\lceil\frac{m}{2}\right\rceil$.
We get two sub-cases c and d:
<u>Sub-case c:</u> if i or $j < \left\lceil\frac{m}{2}\right\rceil$, without loss generality let $i < \left\lceil\frac{m}{2}\right\rceil$, then $f(v_i) = u_i = (p^i)$ since $i+j \geq m$ we get $j > \left\lceil\frac{m}{2}\right\rceil$, therefor $f(v_j) = (p^{j-1}) = u_k$,where $j = \left\lceil\frac{m}{2}\right\rceil+1,\ldots, m-1$ and $k=j-1$ on the other hand u_iu_k adjacent iff $i+k \geq m-1$, because $p^{m-1} \equiv 0 \mod m-1$, therefor u_i adjacent with u_k that mean if e is an edge in $AG(z_{p^m})$, then f(e) is an edge in G.
<u>Sub-case d :</u> if i and $j > \left\lceil\frac{m}{2}\right\rceil$, then $f(v_i) = u_i = (p^{i-1})$ and $f(v_j) = u_j = (p^{j-1})$, since $i > \left\lceil\frac{m}{2}\right\rceil$ then u_i, u_j adjacent where $i \neq j$ and $i,j = \left\lceil\frac{m}{2}\right\rceil, \ldots, m-1$ that is means, if e is an edge in $AG(Z_{p^m})$, then f(e) is an edge .
So that $AG(Z_{p^m}) \cong G$ ■

3. On Annihilating- Ideal Graph of $Z_{p^m q}$.

First we give order and size of AG ($Z_{p^m q}$) for all positive number $m \geq 3$.

Theorem 3.1
Let $R=Z_{p^m q}$, then AG(R) has order 2m vertices and the sizes

$$a_1=\begin{cases}\frac{3m^2}{4}, & \text{if} \quad m \text{ even},\\ \frac{3m^2+1}{4}, & \text{if} \quad m \text{ odd}.\end{cases}$$

Proof : The ideals of ring R are $\{(p),(p^2),\ldots,(p^m),(q),(pq),(p^2 q),\ldots,(p^{m-1}q)\}$. So that $AG(Z_{p^m q})$ has 2m vertices ideal.

Now to find size of graph AG ($Z_{p^m q}$), we must find degree of any vertices ideal. Since $p^m q \equiv 0 \mod p^m q$, and p and q are distinct prime, then (p^i) is adjacent with $(p^j q)$ iff $i+j \geq m$, where $0 \leq j \leq m-1$, $1 \leq i \leq m$. Then
$\deg(p^i) = \sum_{j=m-i}^{m-1} 1 = m-1-(m-i)+1 = i$. Also (q) adjacent with only ideal vertex (p^m) so that $\deg(q) = 1$. Finally since $(p^i q)$ is adjacent with (p^j) iff $i+j \geq m$ and $\sum_{j=m-i}^{m} 1 = i+1$. Also $(p^i q)$ is adjacent with $(p^j q)$ iff $i+j \geq m$, for all $1 \leq i,j \leq m-1$ and since $\sum_{j=m-i}^{m-1} 1 = i$, then $\sum_{j=m-i}^{m-1} 1 + \sum_{j=m-i}^{m} 1 = 2i+1$, also, we note $(p^i q)$ contained loop if and only if $i \geq \left\lceil\frac{m}{2}\right\rceil$
Therefor

$$\deg(p^i q)=\begin{cases}2i+1, & \text{if } 1 \leq i < \left\lceil\frac{m}{2}\right\rceil,\\ 2i, & \text{if } \left\lceil\frac{m}{2}\right\rceil \leq i \leq m-1,\end{cases}$$

Which implies that

$$\sum_{v\in AG(R)} \deg v = 1 + \sum_{i=1}^{\lceil \frac{m}{2} \rceil -1} (2i+1) + \sum_{i=\lceil \frac{m}{2} \rceil}^{i=m-1} 2i + \sum_{i=1}^{i=m} i$$

$$= 1 + \left\lceil \frac{m}{2} \right\rceil - 1 + \sum_{i=1}^{i=m-1} 2i + \sum_{i=1}^{i=m} i \;= \left\lceil \frac{m}{2} \right\rceil + \frac{2(m-1)m}{2} + \frac{m(m+1)}{2}$$

$$= \left\lceil \frac{m}{2} \right\rceil + \frac{3m^2-m}{2}\ .$$

Since

$$\sum_{v\in AG(R)} \deg v = 2a_1$$

, where a_1 is the number of edges of $AG(Z_{p^m q})$, then

$$a_1 = \frac{3m^2}{4}, \text{ if m even}$$

and

$a_1 = \frac{3m^2+1}{4}$, if m odd . ■

Theorem 3.2

for all positive number m≥3. $AG(Z_{p^m q})$ contains a sub-graph $K_{\lceil \frac{m}{2} \rceil +1}$ and $\omega(AG) = \left\lceil \frac{m}{2} \right\rceil + 1$

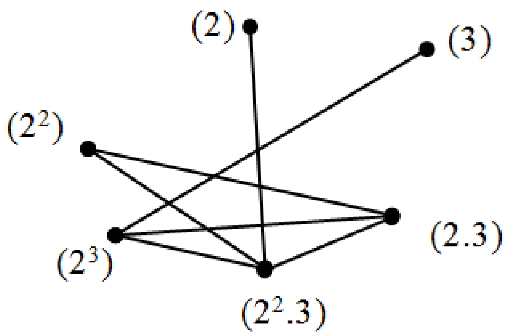
Proof: We can write $V=\{ v_1, v_2, \ldots, v_{\lceil \frac{m}{2} \rceil} \}$, where $v_i=(p^{m-i}q)$, $1 \le i \le \left\lceil \frac{m}{2} \right\rceil$ this mean that all ideal vertices of V are adjacent each other, and V form $K_{\lceil \frac{m}{2} \rceil}$,since (p^m) is adjacent with all vertices in V and not adjacent with (p^i), where $1 \le i \le m-1$, also $v_{\lceil \frac{m}{2} \rceil}$ not adjacent with every vertices $(p^j q)$, where $1 \le j \le \left\lceil \frac{m}{2} \right\rceil - 1$, therefor $V \cup \{(p^m)\}$ form the largest sub-graph $K_{\lceil \frac{m}{2} \rceil +1}$ in $AG(Z_{p^m q})$. Then $\omega(AG(Z_{p^m q})) = \left\lceil \frac{m}{2} \right\rceil + 1$. ■

Proposition 3.3

For all positive number m≥3, we have $diam(AG(Z_{p^m q}))=3$

Proof: In $AG(Z_{p^m q})$ we can find four ideal vertices are defined as $\alpha_1=(p)$, $\alpha_2=(p^{m-1}q)$, $\alpha_3=(p^m)$ and $\alpha_4=(q)$ since $\deg(\alpha_1)=\deg(\alpha_4)$, $\alpha_1\alpha_2=0$, $\alpha_2\alpha_3=0$ and $\alpha_3\alpha_4=0$ but $\alpha_1\alpha_3 \neq 0$, $\alpha_2\alpha_4 \neq 0$ and $\alpha_1\alpha_4 \neq 0$ so that $d(\alpha_1, \alpha_4) = 3$, Therefor dim $AG(Z_{p^m q}) = 3$. ■

Example: Let $R=Z_{24}$, then the diameter of the graph AG(R) is equal 3.



4. On Annihilating - Ideal Graph of $Z_{p^m q^r}$

Let m and r are positive numbers such that m,r ≥ 2 . In this section, we can extended all results in section 3

Theorem4.1

Let $R = Z_{p^m q^r}$, then AG(R) has order mr+m+r-1 vertices and the sizes.

$$a_1 = \begin{cases} \frac{m^2r^2+3m^2r+3mr^2-4m-4r+2r^2+2m^2}{8}, & \text{if m and r are even,} \\ \frac{m^2r^2+3m^2r+3mr^2-3m-3r+2r^2+2m^2+3}{8}, & \text{if m and r are odd,} \\ \frac{m^2r^2+3m^2r+3mr^2-4m-3r+2r^2+2m^2+2}{8}, & \text{if m odd and r even.} \end{cases}$$

Proof : Since R has ideals (p^i), (q^k) where $1 \le i \le m$, $1 \le k \le r$ and (p^iq^k), where $1 \le i \le m$, $1 \le k \le r$, since $p^mq^r \equiv 0 \pmod{p^mq^r}$, hence the order of AG(R) = mr+m+r-1.

Now to find the size we must find degree of all vertices ideal of R since (p^i) adjacent with (p^jq^r) iff $i+j \ge m$, then $\deg(p^i) = \sum_{j=m-i}^{m-1} 1 = i$ similarly $\deg(q^k) = k$.

To find deg (p^iq^k), where $1 \le i \le m$, $1 \le k \le r$ and $(p^iq^k) \ne p^mq^r$.

Since (p^iq^k) adjacent with (p^jq^s) iff $i+j \ge m$ and $k+s \ge r$, so that j= m-i ,…,m and s = r-k ,…,r , which implies that $\sum_{j=m-i}^{m}\sum_{s=r-k}^{r} 1 = (m-(m-i)+1)(r-(r-k)+1) = (i+1)(k+1)$, but $p^mq^r \equiv 0 \pmod{p^mq^r}$, also (p^iq^k) has loop iff $i \ge \left\lceil \frac{m}{2} \right\rceil$ and $k \ge \left\lceil \frac{r}{2} \right\rceil$.

So that $\deg(p^iq^k) = \begin{cases} (i+1)(k+1)-2, & \text{if } \left\lceil \frac{m}{2} \right\rceil \le i \text{ and } \left\lceil \frac{r}{2} \right\rceil \le k, \\ (i+1)(k+1)-1, & \text{otherwise.} \end{cases}$

Now, we find a_1 of R Since $2a_1 = \sum_{v \in AG(R)} \deg v$, then we have three cases

<u>Case 1:</u> If m and r are even

$$2a_1 = \sum_{v \in AG(R)} \deg V = \sum_{i=1}^{m}\sum_{k=1}^{r}((i+1)(k+1)-1) - \sum_{i=\frac{m}{2}}^{m}\sum_{k=\frac{r}{2}}^{r} 1 - ((m+1)(r+1)-2) + \sum_{i=1}^{m} i + \sum_{k=1}^{r} k$$

$$= \sum_{i=1}^{m}(i+1)\sum_{k=1}^{r}(k+1) - \sum_{i=1}^{m}\sum_{k=1}^{r} 1 - \sum_{i=\frac{m}{2}}^{m}\sum_{k=\frac{r}{2}}^{r} 1 - ((m+1)(r+1)-2) + \sum_{i=1}^{m} i + \sum_{k=1}^{r} k$$

$$= \left(\frac{m(m+1)}{2} + m\right)\left(\frac{r(r+1)}{2} + r\right) - mr - \frac{mr+2m+2r+4}{4} - (mr+m+r-1) + \frac{m(m+1)}{2} + \frac{r(r+1)}{2}$$

$$= \frac{m^2r^2+3m^2r \quad +3m \quad r^2-6m-6r+2r^2+2r+2m^2+2m}{4}$$

So that

$$a_1=\frac{m^2r^2+3m^2r+3mr^2\text{-}4m\text{-}4r+2r^2+2m^2}{8}$$

Case 2: If m and r are odd we get

$$2a_1=\sum_{v\in AG(R)}\deg V=\sum_{i=1}^{m}(i+1)\sum_{k=1}^{r}(k+1)-\sum_{i=1}^{m}\sum_{k=1}^{r}1-\sum_{i=\frac{m+1}{2}}^{m}\sum_{k=\frac{r+1}{2}}^{r}1-\big((m+1)(r+1)-2\big)+\sum_{i=1}^{m}i+\sum_{k=1}^{r}k$$

$$=\left(\sum_{i=1}^{m}i+\sum_{i=1}^{m}1\right)\left(\sum_{k=1}^{r}k+\sum_{k=1}^{r}1\right)-rm-\left(\frac{r+1}{2}\right)\left(\frac{m+1}{2}\right)-\big((m+1)(r+1)-2\big)+\sum_{i=1}^{m}i+\sum_{k=1}^{r}k$$

$$=\frac{m^2r^2+3m^2r\ \ +3m\ \ r^2\text{-}5m\text{-}5r+3+2r^2+2r+2m^2+2m}{4}$$

So that

$$a_1\ \ =\frac{m^2r^2+3m^2r+3mr^2\text{-}3m\text{-}3r+2r^2+2m^2+3}{8}$$

Case 3: Since p and q are distinct prime number, then if m odd and r even or r odd and m even we get the same result, so without loss generality let m odd and r even.

$$2a_1=\sum_{v\in AG(R)}\deg V=\sum_{i=1}^{m}(i+1)\sum_{k=1}^{r}(k+1)-\sum_{i=1}^{m}\ \sum_{k=1}^{r}1-\sum_{i=\frac{m+1}{2}}^{m}\ \sum_{k=\frac{r}{2}}^{r}1-\big((m+1)(r+1)\text{-}2\big)+\sum_{i=1}^{m}i+\sum_{k=1}^{r}k$$

$$\left(\sum_{i=1}^{m}i+\sum_{i=1}^{m}1\right)\left(\sum_{k=1}^{r}k+\sum_{k=1}^{r}1\right)\text{-}rm\text{-}\left(\frac{m+1}{2}\right)\left(\frac{r+2}{2}\right)\text{-}(m+1)(r+1)\text{-}2+\sum_{i=1}^{m}i+\sum_{k=1}^{r}k$$

$$=\frac{m^2r^2+3m^2r\ \ +3m\ \ r^2\text{-}6m\text{-}5r+2+2r^2+2r+2m^2+2m}{4}$$

So that

$a_1=\frac{m^2r^2+3m^2r+3mr^2\text{-}4m\text{-}3r+2r^2+2m^2+2}{8}$. ■

Proposition 4.2

For all positive numbers m,r≥2, $diam(AG(Z_{p^mq^r}))=3$.

Proof: By the same method of proof Proposition 3.3, we can choose $\alpha_1=(p)$ and $\alpha_2=(q)$ and we get $d(\alpha_1,\alpha_2)=3$. ■

Theorem 4.3

Let $R=Z_{p^mq^r}$, then AG(R) has a sub-graph of K_s , Furthermore ω(AG (R)=s, where

$$s=\begin{cases}\dfrac{mr+2m+2r}{4}\ , & \text{if m and r are even ,}\\ \dfrac{mr+m+r+1}{4}\ , & \text{if m and r are odd ,}\\ \dfrac{mr+m+2(r+1)}{4}\ , & \text{if m even and r odd.}\end{cases}$$

Proof: If m and r are even, then $v_{ij}=(p^iq^j)$ vertices, where $m/2\leq i\leq m$, $r/2\leq j\leq r$ and $v_{ij}\neq(p^mq^r)=0$ are all adjacent with every others. And the number of this vertices equal $\sum_{m/2}^{m}\sum_{r/2}^{r}1=\frac{mr+2m+2r}{4}=s$.

.Also any vertices in $AG(R)/\{v_{ij}\}$ are non-adjacent with $(p^{m/2}q^{r/2})$, therefore K_s the largest sub-graph in AG(R) in this case. Then $\omega(AG(R))=s$.

If m and r are odd, similarly $v_{ij}=(p^iq^j)$, where $(m+1)/2\leq i\leq m$, $(r+1)/2\leq j\leq r$ and $v_{ij}\neq(p^mq^r)=0$ with the vertices $(p^{(m+1)/2}q^{(r-1)/2})$ and $(p^{(m-1)/2}q^{(r+1)/2})$ the largest sub-graph in AG(R). And the number of vertices equal

$$s=\frac{mr+m+r+1}{4}.$$

.

Similarly, if m even and r odd, we get

$$s=\frac{mr+m+2(r+1)}{4}.$$ ■

5. Hosoya polynomial and Wiener index of Annihilating -Ideal graph of Z_n.

"Hosoya polynomial of the graph G is defined by : $H(G;x) = \sum_{k=0}^{diam(G)} d(G,k)x^k$, where $d(G,k)$ the number of pairs of vertices of a graph G are at distance k a part , for k = 0,1, …, diam(G) . The Winer index of G is define as the sum of all distances between vertices of the graph and denoted by W(G) , and we can find this index by differentiating Hosoya polynomial with respect to x then putting x=1", see[5] ,[8].

In [1] Ahmadi and Jahani-Nezhad first study the Winer index of zero divisor graph of Z_n where $n=p^2$ and p^2q. In [6] Mohammad and Authman extended this result for $n=p^m$ and p^mq and study the Hosoya polynomial of this type. In this section we study Hosoya polynomial and Winer index of annihilating-ideal graph of Z_n, where $n=p^m,p^mq$ and p^mq^r .

Lemma 5.1[5]

Let G be a connected graph of order r. Then $\sum_{i=0}^{diam(G)} d(G,i) = \frac{1}{2}r(r+1)$.

Clearly if $R=Z_{p^2}$, then $AG(R) =K_1$, therefore H (AG(R,x)=1. Also if $R=Z_{p^s}$, where s=3,4, then $AG(R)=K_2$ or $K_{1,2}$ respectively so that $H(AG(R,x)=2+x$ or $3+2x+x^2$ respectively. Therefore we calculate Hosoya polynomial of AG (Z_{p^m}) for all positive number m≥4.

Theorem 5.2

For all positive number m≥4, $H(AG(Z_{p^m},x) = a_0+ a_1x +a_2x^2$, where $a_0= m-1$,

$$a_1=\begin{cases}\frac{m(m-2)}{4}, & \text{if m even,}\\ \frac{(m-1)^2}{4}, & \text{if m odd,}\end{cases}$$ and

$$a_2=\begin{cases}\frac{(m-2)^2}{4}, & \text{if m even,}\\ \frac{m^2-4m+3}{4}, & \text{if m odd.}\end{cases}$$

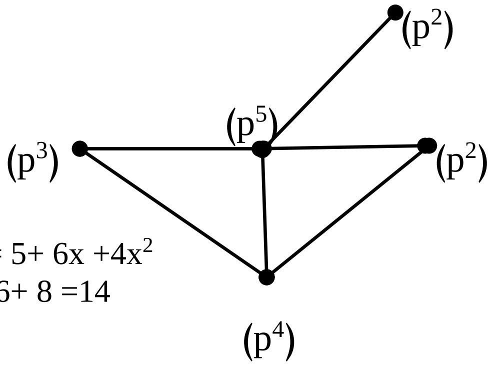
Proof: From Theorem2.1 and Lemma5.1, we get the result. ■

Corollary 5.3

For all positive number m≥4, $W(AG(Z_p^m)) = \begin{cases} \frac{(m-2)(3m-4)}{4}, & \text{if m even,} \\ \frac{(m-1)(3m-7)}{4}, & \text{if m odd.} \end{cases}$

Example 1:

Let R $=Z_p^6$ then



$H(AG(Z_p^6, x) = 5+ 6x +4x^2$

W ($AG(Zp^6)$**)** = 6+ 8 =14

Theorem 5.4

For all positive number m≥3, we have

$$H(AG(Z_{p^m q}, x) = \begin{cases} 2m+\frac{3m^2}{4}x+\frac{5m^2-8m+4}{4}x^2+(m-1)x^3, & \text{if m even,} \\ 2m+\frac{3m^2+1}{4}x+\frac{5m^2-8m+3}{4}x^2+(m-1)x^3, & \text{if m odd.} \end{cases}$$

Proof: Since diam($AG(Z_p{}^m{}_q)$)=3 and applying Theorem3.1, we have a_0= 2m and .

$$a_1 = \begin{cases} \frac{3m^2}{4}, & \text{if m even,} \\ \frac{3m^2+1}{4}, & \text{if m odd.} \end{cases}$$

Now to find a_3 we can write $AG(Z_p{}^m{}_q) = \bigcup_{i=1}^{m}(B_i \cup C_i)$, where $B_i = (p^{m-i}q)$, i=1,2,…,m and $C_i = (p^{m-i+1})$, i=1,2,…,m . Then there are three cases.

Case 1: Let x∈ B_i and y ∈ B_j where 1≤ i ,j ≤ m then $C_1 = (P^m)$ is adjacent with every vertices in B_i 1≤ i ≤m , and that means d(x,y)≤ 2 and this contradict our hypothesis.

Case 2: Let x ∈ C_i and y ∈ Cj where 1≤ i ,j ≤ m we see that a vertex $B_i= (p^{m-1}q)$ is adjacent with every vertices in C_i for all 1≤i≤m because 1+i ≤ 1+m for any 1≤ i ≤m , and that means d(x,y) ≤ 2 and this contradiction.

Case 3: If x ∈ B_i and y ∈ C_j for some 1≤ i ,j ≤ m , in this case we see that d(x,y) =3 if and only if i=m and 2 ≤ j ≤ m , because that d(x,y) ≤ 2 for any 1≤ i ≤ m-1 and 2 ≤ j ≤ m also d(x,y) = 1 for 1 ≤ i ≤ m and j= 1 , therefor the number of of pairs of vertices that that are distance three apart is (m-1).

Finally, we find a_2, applying by Lemma 5.1 we get

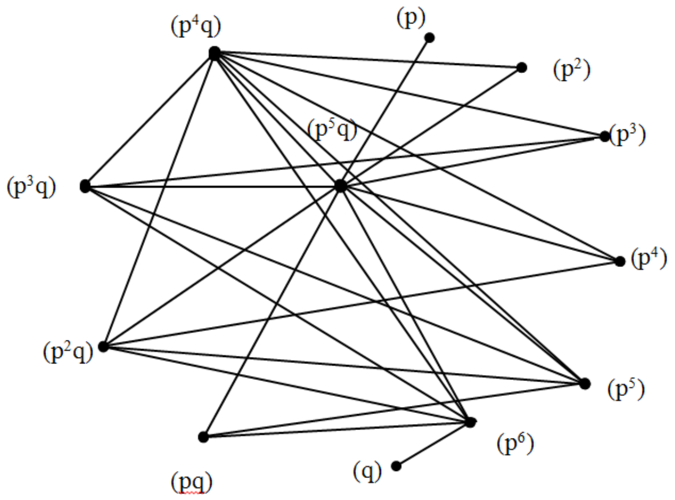
$a_2 = (5m^2- 8m+4)/4$, when m even, $a_2 = (5m^2-8m+3)/4$, when m odd.■

Corollary 5.5:

$$W\left(AG\left(Z_{p^m q}\right)\right) = \begin{cases} \frac{13\,m^2-4m-4}{4}, & \text{if m even,} \\ \frac{13\,m^2-4m-5}{4}, & \text{if m odd.} \end{cases}$$

Example 2:

Let $R=Z_P{}^6{}_q$, then $H(AG(Z_P{}^6{}_q,x)) = 12+ 27x +34x^2+ 5x^3$, $W(AG (Z_P{}^6{}_q))$=27+2.34+3.5=110



Finally we give extended to theorem 5.4

Theorem 5.6

For any positive numbers m,r≥2, we have

$H(AG(Z_{p^m q^r}))=a_0+a_1x+a_2x^2+a_3x^3$, where $a_0=mr+m+r-1$ and

$$a_1=\begin{cases}\dfrac{m^2r^2+3m^2r+3mr^2-4m-4r+2r^2+2m^2}{8}, & \text{if m and r are even,}\\ \dfrac{m^2r^2+3m^2r+3mr^2-3m-3r+2r^2+2m^2+3}{8}, & \text{if m and r are odd,}\\ \dfrac{m^2r^2+3m^2r+3mr^2-4m-3r+2r^2+2m^2+2}{8}, & \text{if m odd and r even,}\end{cases}$$

$$a_2=\begin{cases}\dfrac{3m^2r^2+5m^2r+5mr^2+2m^2+2r^2-12mr-8m-8r+16}{8}, & \text{if m and r even,}\\ \dfrac{3m^2r^2+5m^2r+5mr^2+2m^2+2r^2-12mr-9m-9r+13}{8}, & \text{if m and r odd,}\\ \dfrac{3m^2r^2+5m^2r+5mr^2+2m^2+2r^2-12mr-8m-9r+14}{8}, & \text{if m odd and r even.}\end{cases}$$

and $a_3= mr-1$

Proof: By Theorem 4.1, AG(R) has order $a_0=mr+m+r-1$ vertices and the sizes.

$$a_1=\begin{cases}\dfrac{m^2r^2+3m^2r+3mr^2-4m-4r+2r^2+2m^2}{8}, & \text{if m and r are even,}\\ \dfrac{m^2r^2+3m^2r+3mr^2-3m-3r+2r^2+2m^2+3}{8}, & \text{if m and r are odd,}\\ \dfrac{m^2r^2+3m^2r+3mr^2-4m-3r+2r^2+2m^2+2}{8}, & \text{if m odd and r even,}\end{cases}$$

To find a_3 let A= $\{(p^iq^j): 1\leq i\leq m$ and, $1\leq j\leq r\}-\{0\}$, B $=\{(p^i): 1\leq i\leq m\}$ and C = $\{(q^j): 1\leq j\leq r\}$.

If x,y ∈ A or B , since every element in this case is adjacent with ideal vertex $(p^{m-1}q^r)$ so that $d(x,y)\leq 2$ which is contradiction .

Similarly , if x,y ∈ A or C ,then every element in this case is adjacent with ideal vertex (p^mq^{r-1}) so that $d(x,y)\leq 2$ which is contradiction, if x∈ B and y ∈ C since every element in B is adjacent with ideal vertex ($p^i q^r$) for i = 1,…, m-1 and therefor d(x,y) = 3 except the case where i=m and j=r so that the number of pairs of vertex that are distance three a part is mr-1 .

To find a_2, since $\sum_{i=0}^{diam\,(G)} d(G,i) = \frac{r(r+1)}{2}$ by Lemma 5.1, then

$$a_2=\begin{cases} \frac{3m^2r^2+5m^2r+5mr^2+2m^2+2r^2-12mr-8m-8r+16}{8}, & \text{if m and r even,} \\ \frac{3m^2r^2+5m^2r+5mr^2+2m^2+2r^2-12mr-9m-9r+13}{8}, & \text{if m and r odd,} \\ \frac{3m^2r^2+5m^2r+5mr^2+2m^2+2r^2-12mr-8m-9r+14}{8}, & \text{if m odd and r even.} \end{cases}$$

Corollary 5.7:

$$W\left(AG(Z_{p^m q^r})\right)=\begin{cases} \frac{7\,m^2r^2-20m-20r+13m^2r+13mr^2+6r^2+6m^2+8}{8} & \text{if m and r even} \\ \frac{7\,m^2r^2-21m-21r+13m^2r+13mr^2+6r^2+6m^2+5}{8} & \text{if m and r odd} \\ \frac{7\,m^2r^2-20m-21r+13m^2r+13mr^2+6r^2+6m^2+6}{8} & \text{if m odd and r even} \end{cases}$$

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