On Rings whose Maximal Ideals are GP-Ideals

Raida D. Mahmood **Manal A. Abd**
raida.1961@uomosul.edu.iq
College of Computer Sciences and Mathematics
University of Mosul, Iraq

Received on: 28/08/2004 **Accepted on: 05/04/2005**

ABSTRACT

This paper introduces the notion of maximal GP-ideal .We studied the class of rings whose maximal left ideal are right GP-ideal. We call such ring MRGP-rings. We consider a necessary and sufficient condition for MRGP-rings to be MRCP-rings. We also study the connection between MRGP-ring, kasch ring, division ring and the strongly regular ring.

Key words: strongaly regular , kasch , GP-ideal , division ring.

حول الحلقات التي فيها كل مثالي اعظمي من النمط – GP

د.رائدة داؤد محمود **منال ادريس عبد**
كلية علوم الحاسوب والرياضيات
جامعة الموصل

تاريخ الاستلام: 2004/08/28 **تاريخ القبول: 2005/04/05**

الملخص

يقدم هذا البحث تحديدا لمفهوم المثاليات المعممة العظمى وتمت دراسة الحلقات التي تكون فيها المثاليات العظمى معممة. كما تم تعريف هذه الحلقات على أنها من النمط MRGP وتم عرض بعض خواصها وعلاقتها مع الحلقة المنتظمة بقوة ،حلقة كاش وحلقة القسمة.

الكلمات المفتاحية: الحلقة المنتظمة بقوة , حلقة كاش ,المثاليات المعممة , حلقة القسمة.

1- Introduction:

Throughout this paper, R will denote an associative ring with identity. For any element a in R, we define the right annihilater of a by $r(a)=\{x\in R:ax=0\}$, and likewise the left annihilater l(a) . Y,Z,J will denote respectively the right singular ideal, the left singular ideal and the Jacobson radical of R. Recall that 1) An ideal I is said to be a right (left) pure if for every $a\in I$, there exists $b\in I$ such that a=ab(ba) [1] 2) R is called a uniform ring if for every non-zero ideal of R is essential , [4] .3)
A ring R is said to be left kasch ring, if every maximal right ideal is a right annihilator [3] 4) R is said to be strongly regular if for each $a\in R$, there exists $x\in R$ such that $a=a^2x$. Following [1] . 5)A ring R is called reduced if R has no non-zero nilpotent element and an ideal I of a ring R is said to be right (left) GP-ideals if for every $a\in I$, there exists $b\in I$ and a positive integer n such that $a^n=a^nb(ba^n)$ [5].

2-MRGP-Rings:

Following [6] a maximal left ideal M of the ring R is said to be a right Co-pure if for every $a\in M$, Ma is a right pure.

Definition 2-1: A ring R is called MRCP-ring , if all maximal left ideals are right Co-pure , .see[6]

Definition 2-2: R is called MRGP-ring , if for any maximal left ideal M of R, any $a\in M$, Ma is a right GP-ideal .

Clearly every MRCP-ring is an MRGP-ring, however the converse is not true as the following example shows:
Example: The ring Z_{12} of integers modulo 12 is an MRGP-ring but not an MRCP-ring.

We now consider a necessary and sufficient condition for MRGP-ring to be an MRCP-ring.

Theorem 2-3: Let R be a reduced MRGP-ring. Then R is MRCP-ring

Proof: Let M be any maximal left ideal of R. Since R is MRGP-ring , then Ma is a right GP-ideal of R and there exists c, b $\in M$ and a positive integer n such that $(ba)^n=(ba)^n(ca)$ this implies that $(ba)^n(1-ca)=0$ and hence $(1-ca)\in r(ba)^n\subseteq r(ba)$.

Therefore $(1-ca)\,)\in r(ba)$, whence $(ba)(1-ca)=0$. Thus ba=ba ca (Ma is a right pure) and hence R is MRCP-ring.

Lemma 2-4: let a be a non-zero element of the ring R and let r(a) =0. Then $r(a^{2n})=0$

Theorem 2-5: let R be an MRGP-ring and r(a)=0 . Then a is a left invertible.

Proof: let $a \in R$ with r(a)=0 . If $Ra \neq R$, there exists a maximal left ideal M containing Ra . Since R is an MRGP- ring, there exists $b \in M$ and a positive integer n such that $a^{2n}=a^{2n}(ba)$ .Whence $(1\text{-}ba) \in r(a^{2n})=0$, yielding $1 \in M$, which contradicts $M \neq R$. Therefore Ra=R. In particular ra=1 for some $r \in R$. Hence a is a left invertible.

Proposition 2-6: Let R be a right uniform reduced MRGP-ring. Then R is a division ring.

Proof: If $0 \neq a \in R$ and $Ra \neq R$. Let M be a maximal left ideal containing Ra. Since R is MRGP-rings , then for every $a \in M$, Ma is a right GP-ideal, so $(ca)^n=(ca)^n ba$, for some $c,b \in M$ and a positive integer n. Since R is a right uniform then every right ideal is essential ideal.

Consider $l(ba) \cap R(ca)^n$, let $x \in l(ba) \cap R(ca)^n$ implies that xba=0 and $(ca)^n=x$,so $(ca)^n ba=0$, then $(ca)^n ba=(ca)^n=0$.Therefore $l(ba) \cap R(ca)^n=0$ implies l(ba)=0, since R is reduced , r(ba)=0 .By Theorem (2-5) , ba is a left invertible , there exists $y \in R$ such that v(ba)=1 , so $(vb)a=1 \in M$, a contradiction. Therefore Ra=R. So R is division ring.

A ring R is called zero commutative (briefly ZC) [2] if for $a,b \in R$ ab=0 implies ba=0

Proposition(2-7): Let R be a zero commutative , MRGP-ring. Then R is a kasch ring .

Proof: Let M be any maximal left ideal of R, and let Z be the left singular ideal of R, if $M \cap Z=0$, then for any $y \in Z$, $y \notin M$, this implies that l(y) is an essential left ideal of R.

Let $x \in l(y) \cap l(1\text{-}y)$, then xy=0 and x(1-y)=0 , yields x=xy=0. Therefore $l(y) \cap l(1\text{-}y)=0$, whence l(1-y)=0.

Since R is a zero commutative , then we have r(1-y)=0 . By Theorem (2-5) 1-y is an invertible element of R , hence $y \in J \subset M$ a contradiction. Thus $M \cap Z \neq 0$.

Let $a \in M \cap Z$, since R is an MRGP-ring , then Ma is a right GP-ideal of R and there exists $c,b \in M$ and a positive integer n such that

$(ba)^n=(ba)^n(ca)$. We claim that $l(ca) \cap R(ba)^n=0$, if not let $d\in l(ca)\cap R(ba)^n$, then d $ca=0$ and $d=r(ba)^n$ for some r in R,so $r(ba)^n ca=0$,implies $r(ba)^n =0$, whence $d=0$. Therefore $l(ca) \cap R(ba)^n=0$, but $r(ca)$ is essential then $R(ba)^n=0$ and hence $b^n a^n=0$ implies that $b^n\in l(a^n)$ and $a^n\in r(b^n)$ .Therefore $M=l(a^n)$ and $M=r(b^n)$.

The following theorem gives the condition of being MRGP-ring is strongly regular.

<u>Theorem 2-8:</u> Let R be a reduced MRGP-ring. Then R is strongly regular.

<u>Proof</u>: Let z be a non-zero element in R. We claim that $Rz+l(z)=R$. If $Rz+l(z)\neq R$, let M be a maximal left ideal containing $Rz+l(z)$. Since R is an MRGP- ring, then Mz is a right GP-ideal and there exists $c\in M$ and a positive integer n such that $(z)^{2n}=(z)^{2n}(cz)$. Whence $(1-cz)\in r(z)^{2n}$. Since R is reduced, we have $(1-cz))\in r(z)^{2n}=l(z^{2n})=l(z)\subseteq M$ this implies that $1\in M$, a contradiction .Therefore $Rz+l(z)=R$, in particular $xz+y=1$, $x\in R$, $y\in l(z)$.

Thus $z=xz^2$, and therefore R is strongly regular.

FERENRECES

[1] Al-Ezeh H. (1989), "Pure Ideals in Commutative Reduced Gelfand Ring with unity" , **Arch.** Vol.53.266-269.

[2] Nam S.B, kim N.K. and Kim. J.Y. (1999), "On Simple Singular GP-injective Modules", **Comm. In Algebrn** 27,(5).

[3] Mohammand M.A.(2003) , **Kasch Rings ,** M.Sc Thesis. Mosul Univ.

[4] Ming. Y.C. (1983), "Maximal ideals in Regular Rings"**, Hokk-Math. J.** 12,119-128.

[5] Shuker N.H. and Raida D.M. (2000), "On Generalization of Pure Ideals ", **J. Edu. And Sci.** Vol (43), 86-89.

[6] Shuker N.H. and Raida D.M. (2000), "On Rings Whose Maximal Ideals are Co-Pure", **J. of Science Raf.** Vol. 11.,No.(4).