Zero Divisor Graph Of $Z_{p^m qr}$ with Applications

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ABSTRACT

In this paper, we study zero-divisor graph of the ring $Z_{p^m qr}$ and give some properties of this graph. Also, we find the chromatic number, Hosoya polynomial and Wiener index of this graph .

Keywords: zero-divisor graph, chromatic number, Hosoya polynomial, Wiener index.

بيان قواسم الصفر لـ $Z_{p^m qr}$ مع التطبيقات

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الملخص

في هذا البحث درسنا بيان قواسم الصفر للحلقة $Z_{p^m qr}$ واعطينا بعض الخواص لهذا البيان كذلك وجدنا العدد اللوني ومتعدد حدود هوسويا ودليل وينر لهذا البيان.

الكلمات المفتاحية: بيان قواسم الصفر، العدد اللوني، متعدد حدود هوسويا، دليل وينر.

1. Introduction

For a commutative ring R with identity $1\neq 0$, let Z(R) be the set of zero divisor of R , and $Z(R)^* = Z(R)\backslash\{0\}$. An directed graph $\Gamma(R)$ is a simple graph with vertices in $Z(R)^*=Z(R)-\{0\}$,and edges $E(\Gamma(R))=\{e=uv: u,v\in V(\Gamma(R))$ if and only if $uv=0\}$. Many authors studied this concept see [e.g. 2 and 5] .The distance between a pair of vertices (u,v)of the graph is the length of the shortest path between u,v. The diameter of a connected graph G, denoted diam(G)= d(u,v) : is the maximum distance between two vertices. The eccentricity e(v) of a vertex is the maximum distance from it to any other vertex, that is e(v)=max d(u,v) for all $u,v\in V(G)$, the radius of G is rad (G) = min d(u,v) : for all $u,v\in V(G)$ and the center of G is defined by Cent(G) = $\{x\in V(G) : d(x,y) = rad(G)\}$. $\left\lceil\frac{m}{2}\right\rceil$($\left\lfloor\frac{m}{2}\right\rfloor$ resp.) It means that the smallest integer is not less than $\left\lceil\frac{m}{2}\right\rceil$ (the greatest integer is not greater than $\left\lfloor\frac{m}{2}\right\rfloor$resp.) . For any ring R with $|Z(R)^*| \geq 1$ and $a \in Z(R)^*$, recall that the neighborhood of a in $\Gamma(R)$, denoted by $N_{\Gamma(R)}(x)$, is defined as the set of all b∈ $Z(R)^*$,such that b is adjacent to a in Γ(R). That is, $N_{\Gamma(R)}(x) = \{y\in Z(R)^*\setminus\{x\} | x\, y=0\}$.

A complete sub-graph K_n of a graph G is called a clique , and we denoted ω (G) is the clique number of graph G, which is the greatest integer n ≥ 1such that $K_n\subseteq G$**.** In [9] Hosoya gave the concept of Hosoya polynomials as follows: $H(G,x)= \sum_{k=0}^{diam(G)} d(G;k)x^k$,such that d(G,k) ,k ≥ 0, be the number of vertex pairs at distance k in connected graph G ." The Wiener index of G is the sum of the distance between all pair of vertices of G, that is W(G)=∑d(u,v) ,where u,v∈V ,and we can find this index by differentiating Hosoya polynomial with respect to x then putting x=1, see [8 and 10]" .

In [1] Ahmadi and Jahani-Nazhad studied the Wiener index of graph $\Gamma(Z_{pq})$, $\Gamma(Z_{q^2})$ where p,q are distinct primes .In [10] Mohammed and authman studied the Hosoya polynomials and Wiener index of $\Gamma(Zp^m q)$,$\Gamma(Zp^m)$ and gave the properties of them . In [13] Shuker, Mohammad and Khaleel studied the zero divisor graph of $Z_{p^m q^2}$ and gave Hosoya polynomials and Wiener index of this graph. In this work we extend this result to the zero divisor graph of $Z_{p^n qr}$, where m ≥ 2 and p,q are distinct primes ,and we give new properties of Hosoya polynomials of $\Gamma(Z_{p^m\, q\, r})$.

2. Properties of $\Gamma(Z_{p^m\, q\, r})$

In this section we give som**e** properties of $\Gamma(Z_{p^m\, q\, r})$ also we find chromatic number of this graph **,** where m≥2 and p, q ,r are distinct prime number.

Lemma 2.1 [7]

Let Z_n be a ring of integers modulo n. then the number of all non-zero-divisor for k/n are $\frac{n}{k} - 1$.

First we give classification of vertices of $\Gamma(Z_{p^m\, q\, r})$

2.2: Classification of $V(\Gamma(Z_{p^m\, q\, r}))$

Let $R=Z_{p^m qr}$,we can write

$A_1=(P^{m-1}qr) -\{0\}$,

$A_i= (p^{m-i}qr) -\{(p^{m-i+1} qr)\}$, i=2,3,…,m,

$B_0= (p^m q) -\{0\}$,

$B_j= (p^{m-j} q) - \{(p^{m-i} qr) \cup (p^{m-j+1} q) \}$, j=1,2,3,…,m,

$C_0= (p^m r) -\{0\}$,

$C_k= (p^{m-k}r) - \{(p^{m-k} qr) \cup (p^{m-k+1} r) \}$, k=1,2,3,…,m.

$D_0= (p^m) -\{0\}$,

$D_l= (p^{m-l}) - \{(p^{m-l} q) \cup(p^{m-l} r)\cup(p^{m-l+1}) \}$, l=1,2,3,…,m-1.

Clearly all these sets are disjoint and $Z(R)^*=\bigcup_{i=1}^{m} A_i \bigcup_{j=0}^{m} B_j \bigcup_{k=0}^{m} C_k \bigcup_{l=0}^{m-1} D_l$, and by using lemma 2.1, we can find the cardinality of sets A_i as follows:

$|A_1| = p-1$, $|A_i| = \left(\frac{p^m qr}{p^{m-i}qr} - 1\right) - \left(\frac{p^m qr}{p^{m-i+1}qr} - 1\right) = p^i - p^{i-1}$, i=2,…,m.

Similarly

$|B_0| = r-1$, $|B_j| =(p^j - p^{j-1})(r-1)$, j=1,2,3,…,m.

$|C_0| = q-1$, $|C_k| =(p^k - p^{k-1})(q-1)$, k=1,2,3,…,m .

$|D_0| = (q-1)(r-1)$, $|D_l| = (p^l - p^{l-1}) (q-1)(r-1)$, l=1,2,…,m-1.

Remark 2.3 :

For any non-negative integer number s we get**,**

$\sum_{i=1}^{s}|A_i| =(p-1)+(p^2 - p)+(p^3 - p^2)+\ldots+ (p^s - p^{s-1})= p^s -1$,

$\sum_{j=0}^{s}|B_j|$ =(r-1)+(p – 1)(r-1)+ (p² – p)(r – 1)+…+ ($p^s - p^{s-1}$)(r – 1) =p^s(r-1),
$\sum_{k=0}^{s}|C_k|$=(q-1)+(p – 1)(q-1)+ (p² – p)+(q – 1)+…+ ($p^s - p^{s-1}$)(q – 1) = p^s(q-1),
$\sum_{l=0}^{s-1}|D_l|$=(q-1) (r – 1)+(p – 1)(q-1) (r – 1)+ (p² – p) (q – 1) (r – 1)+…+ ($p^s - p^{s-1}$)(q – 1) (r – 1) = p^{s-1}(q-1) (r – 1) .
Next, we shall give the following results:

Theorem 2.4:

Let R= $Z_{p^m\,q\,r}$ and A_i, B_j, C_k, D_l be subsets as classification 2.2 and x∊ $Z(R)^*$, where i=1,…,m, j=0,…,m, k=0,…,m, l=0,…,m-1 .Then

$$\deg_{\Gamma(Z_{p^m qr})} x = \begin{cases} p^{m-i}qr-2 \ , \ \text{if } x\in A_i \text{ and } 1\le i\le \left\lfloor \frac{m}{2}\right\rfloor . \\ p^{m-i}qr-1 \ , \ \text{if } x\in A_i \text{ and } \left\lfloor \frac{m}{2}\right\rfloor+1\le i\le m . \\ p^{m-j}r-1 \ , \ \text{if } x\in B_j \text{ and } 0\le j\le m. \\ p^{m-k}q-1 \ , \ \text{if } x\in C_k \text{ and } 0\le k\le m . \\ p^{m-l}-1 \ , \ \text{if } x\in D_l \text{ and } 0\le l\le m-1 . \end{cases}$$

Proof :

First let x∊ A_i for 1≤i≤m and let y ∊$Z(R)^*$, then if y∊ A_j, where j=1,2,…,m, since xy=0 (mod p^mqr), if and only if i+j≤ m, then y ∊ $N_{\Gamma(R)}(x)$ if and only if j = 1, … , m-i, since $\sum_{i=1}^{m-i}|A_i|=p^{m-i}-1$, and x ∊ A_i has loop if and only if 1≤ i ≤ $\left\lfloor \frac{m}{2}\right\rfloor$, we have the number of adjacent elements with A_i in this case is p^{m-i}-2, if 1 ≤ i ≤ $\left\lfloor \frac{m}{2}\right\rfloor$, and p^{m-i}-1, if $\left\lfloor \frac{m}{2}\right\rfloor + 1 \le i \le m$.

Similarly if y ∊ B_j (C_k or D_l resp.), then the number of adjacent elements with A_i are $\sum_{j=0}^{m-i}|B_j|$ = p^{m-i}(r-1) ($\sum_{k=0}^{m-i}|C_k|$= p^{m-i}(q-1) or $\sum_{l=0}^{m-i}|D_l|$=p^{m-i}(r-1) (q-1) resp.) . Therefore the degree of the vertex x ϵ A_i , 1 ≤ i ≤ $\left\lfloor \frac{m}{2}\right\rfloor$, is

$$\deg(x)_{x\in A_i} = \begin{cases} p^{m-i}qr-2 \quad \text{if } 1\le i\le \left\lfloor \frac{m}{2}\right\rfloor , \\ p^{m-i}qr-1 \quad \text{if } \left\lfloor \frac{m}{2}\right\rfloor+1\le i\le m . \end{cases}$$

Finally if x ∊ B_j, j=0,1,…,m, and y ∊ $Z(Z_{p^m\,q\,r})^*$, and if y ∊ C_k or y ∊ D_l ,then xy≠0.Hence y ∊ A_i or y ∊ B_k ,with 1 ≤ i ≤ m-j and 0 ≤ k ≤ m-j and by the same method of above proof we have $\deg(x)_{x\in B_j}$= p^{m-i} r-1 , 0≤ j ≤m. Similarly we can find, $\deg(x)_{x\in C_k}$ =p^{m-i} (q-1), 0 ≤ k ≤ m, and $\deg(x)_{x\in D_l}$=p^{m-i} -1**,** 0 ≤ l ≤ m-1.

Theorem 2.5 .

Let R= $Z_{p^m\,q\,r}$ and A_i, B_j, C_k, D_l be a subsets as classification 2.2**,** where, i=1,2,3,…,m, j=0,1,2,3,…,m, k=0,1,…,m-1. Then Cent(Γ(R))=$\bigcup_{i=1}^{m-1} A_i \cup B_0 \cup C_0$

Proof.

Let x ∊ Γ (R) . Then we have the following two cases:

Case1.

If min {e(x)} = 1, then x is adjacent to all other vertices. This mean that R is a local ring or R≅$F_1 \times F_2$, where F_i ,i∊{1,2} is a field but not local **[** 14 **]** , this contradict the fact that R is not local ring or isomorphic with $F_1 \times F_2$.

Case2.

If min {e(x)} =2 or 3 .Since the diam(Γ(R)) ≤ 3, then max {e(x)} ≤ 3 .So min {e(x)}=2 this mean that the eccentricity of the vertices of the center of Γ(R) must be 2 .We observe that $x \in A_i$,$y \in B_0$, $z \in C_0$ where i=1,,2,…,$m-1$, has the eccentricity 2.

Example 1.

Let $\Gamma(Z_{p^m.q.r})$,is graph with p=2,q=3,r=5,m=5,then

$Z(Z_{2^5.3.5})^*$={2,3,4,5,6,8,9,10,12,14,15,16,18,20,21,22,24,25,…,478},

$Z(Z_{2^5.3.5})^*= Z(Z(480))^* = \bigcup_{i=1}^{m} A_i \bigcup_{j=0}^{m} B_j \bigcup_{k=0}^{m} C_k \bigcup_{l=0}^{m-1} D_l$.

$A_1=(2^4.3.5)$-{0}={240}.

$A_2=(2^3.3.5)$-{A_1∪{0}}={120,360}.

$A_3=(2^2.3.5)$-{A_1∪A_2∪{0}}={60,180,300,420} .

A_4=(2.3.5)-{A_1∪A_2∪A_3∪{0}}={30,90,150,210,270,330, 390,450}.

A_5=(3.5)-{A_4∪{0}}={15, 45, 75, 105, 135, 165, 195, 225, 255, 285, 315, 345, 375, 405, 435, 465}.

$B_0=(2^5.3)$-{0}={96,192,288,384}.

$B_1=(2^4.3)$-{B_0∪A_1∪{0}}={48,144,336,432}.

$B_2=(2^3.3)$={B_0∪B_1∪A_1∪A_2∪{0}}={24,72,168,216,264,312,408,456}

$B_3=(2^2.3)$={B_2∪A_3∪{0}}={12, 36, 84, 108, 132, 156, 204, 228, 252, 276, 324, 348, 372, 396, 444, 468}.

B_4=(2.3)={B_3∪A_4∪{0}}={6, 18, 42, 54, 66, 78, 102, 114, 126, 138, 162, 174, 186, 198, 222, 234, 246, 258, 282, 294,306, 318, 342, 354, 366, 378,402 , 414, 426, 438, 462, 474}.

B_5=(3)-{B_4∪A_5∪ {0}}={3, 9, 21, 27, 33, 39, 51, 57, 63, 69, 81, 87, 93, 99, 111, 117, 123, 129, 141,…, 459, 471, 477}.

$C_0=(2^5.5)$-{0}={160, 320}.

$C_1=(2^4.5)$-{C_0∪A_1∪{0}}={80, 400}.

$C_2=(2^3.5)$-{C_1∪A_2∪{0}}={40, 200, 280, 440}.

$C_3=(2^2.5)$-{C_2∪A_3∪{0}}={20, 100, 140, 220, 260, 340, 388, 460}.

C_4=(2.5)-{C_{30}∪A_4∪{0}}={10, 50, 70, 110, 130, 170, 190, 230, 250, 290, 310, 350, 370, 410, 430, 470}.

C_5=(5)-{C_4∪A_5∪{0}}={5, 25, 35, 55, 65, 85, 95, 115, 125, 145, 155, 175, 185, 205, 215, 235, 245, 265, 275, 295, 305, 325, 335, 355, 365, 385, 395, 415, 425, 445, 455, 475}.

$D_0=(2^5)$-{B_0∪C_0∪{0}}={32, 64, 128, 416, 224, 256, 448, 352}.

$D_1=(2^4)$- {D_0 ∪B_1∪C_1∪A_1{0}}={16, 112, 176, 208, 272, 304, 368, 464}.

$D_2=(2^3)$-D_1∪B_2∪C_2∪A_2{0}}={8, 56, 88, 104, 136, 152, 184, 232, 248, 28, 344, 376, 392, 424, 472}.

$D_3=(2^2)$-{D_2∪B_2∪C_2∪A_2{0}}={4, 28, 44, 52, 68, 76, 92, 116, 124, 148, 164, 172, 188, 196, … 436, 452, 476}.

D_4=(2)-{D_3∪B_3∪C_3∪A_3{0}}={2, 14, 22, 26, 34, 38, 46, 58, 62, 74, 82, 86, 94, 98, 106, 118, 122, .. , 458, 466, 478}.

Then the

$\deg(\Gamma(Z_{480}))_{x\in A1}$ =288 , $\deg(\Gamma(Z_{480}))_{x\in A2}$ =118, $\deg(\Gamma(Z_{480}))_{x\in A3}$ =59, $\deg(\Gamma(Z_{480}))_{x\in A4}$ =29, $\deg(\Gamma(Z_{480}))_{x\in A5}$ =14,

$\deg(\Gamma(Z_{480}))_{x\in B0}$ =95, $\deg(\Gamma(Z_{480}))_{x\in B1}$ =47, $\deg(\Gamma(Z_{480}))_{x\in B2}$ =23,

$\deg(\Gamma(Z_{480}))_{x\in B3}$ =11, $\deg(\Gamma(Z_{480}))_{x\in B4}$ =6, $\deg(\Gamma(Z_{480}))_{x\in B5}$ =2, $\deg(\Gamma(Z_{480}))_{x\in C0}$ =159, $\deg(\Gamma(Z_{480}))_{x\in C1}$ =79, $\deg(\Gamma(Z_{480}))_{x\in C2}$ =39

$\deg(\Gamma(Z_{480}))_{x\in C3}$ =19, $\deg(\Gamma(Z_{480}))_{x\in C4}$ =9, $\deg(\Gamma(Z_{480}))_{x\in C5}$ =4

$\deg(\Gamma(Z_{480}))_{x\in D0}=31$, $\deg(\Gamma(Z_{480}))_{x\in D1}=15$, $\deg(\Gamma(Z_{480}))_{x\in D2}=7$
$\deg(\Gamma(Z_{480}))_{x\in D3}=3$, $\deg(\Gamma(Z_{480}))_{x\in D4}=1$, and
$\mathrm{Cent}(\Gamma(Z_{480}))=\bigcup_{i=1}^{m-1} A_i \cup B_0 \cup C_0$.

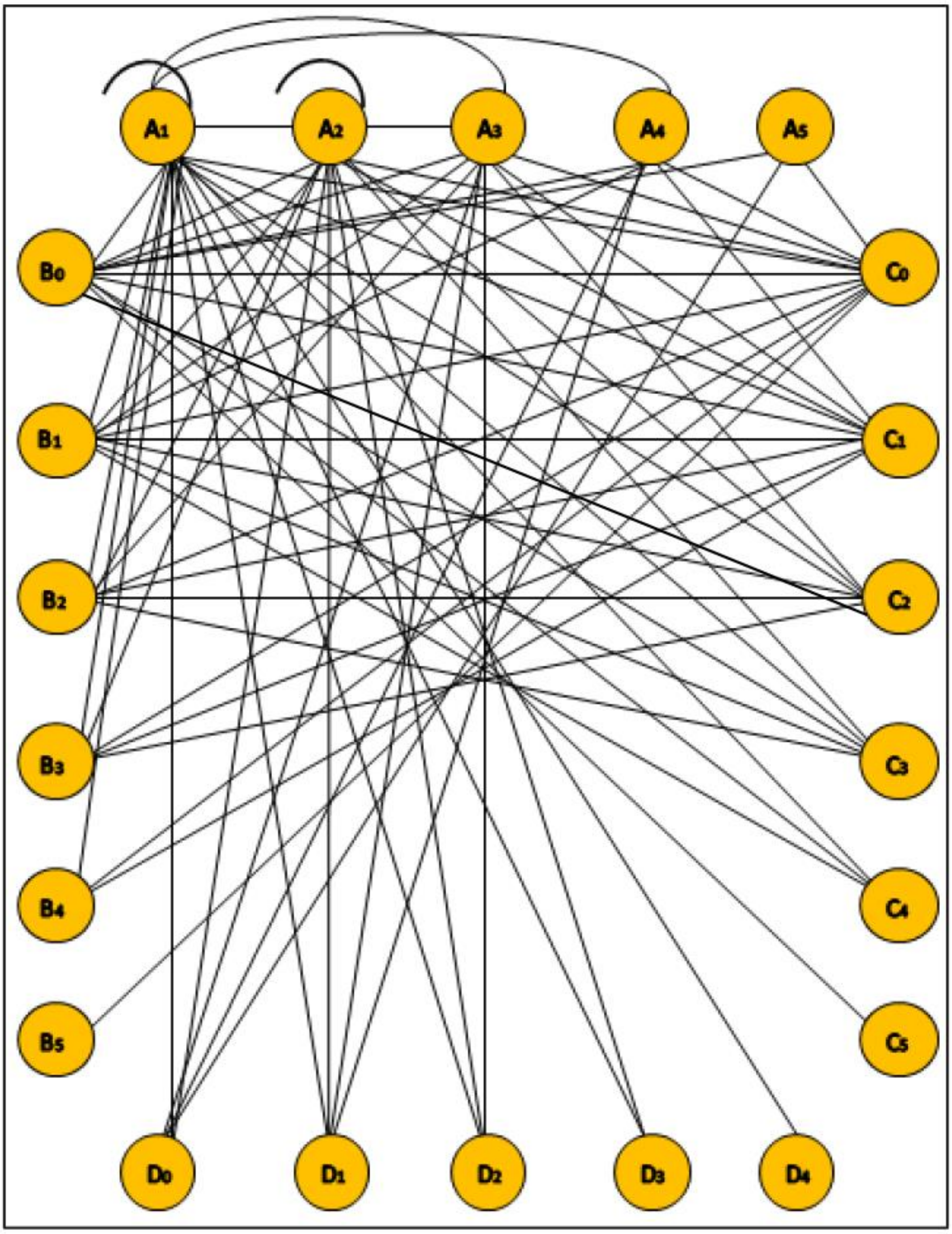

Fig. $\Gamma(Z_{2^5.3.5})$

Theorem 2.6:

Let $R=Z_{p^m\,q\,r}$ and A_i, B_j, C_k, D_l be a subsets as classification 2.2 if $x\in Z(R)^*$, where $i=1,\ldots,m$, $j=0,\ldots,m$, $k=0,\ldots,m$, $l=0,\ldots,m-1$ .Then

$$\omega(\Gamma(Z_{p^m qr}))=\begin{cases} p^{\frac{m}{2}}+1 & ,\text{if } m \text{ even.} \\ p^{\frac{m-1}{2}}+2 & ,\text{if } m \text{ odd }. \end{cases}$$

Proof:

First, if m even number let $J=\bigcup_{i=1}^{\frac{m}{2}} A_i \cup\{x,y\}$, where $x=p^m q \in B_0$,and $y=p^m r \in C_0$ .It is clear that J is a complete sub-graph and $J=K_{p^{\frac{m}{2}}}$.We claim that J is the greats complete sub-graph .Let $z \in \Gamma(R)-J$, then there are three cases:

Case 1: If $z \in B_j$, where j=1,…,m ,then $z \notin N_{\Gamma(R)}(x)$

Case 2: If $z \in C_k$,where k=1,…,m , or D_l where l=1,…,m-1, then $z \notin N_{\Gamma(R)}(y)$.

Case 3: If $z \in A_i$, where $\frac{m}{2} < i \le m-1$, then $z \notin N_{\Gamma(R)}(w)$ for any $w \in A_{\frac{m}{2}}$. Then J the greats complete sub-graph of Γ(R). Therefore $\omega(\Gamma(R))=\sum_{i=1}^{\frac{m}{2}}|A_i|+2=p^{\frac{m}{2}}+1$.

Second if m odd let $J=\bigcup_{i=1}^{\frac{m-1}{2}} A_i \cup\{x,y,z\}$, where $x=p^m q$, $y=p^m r$ and $z=p^{\frac{m-1}{2}}qr$, then by the same method of above prove we have J the greats complete sub-graph of Γ (R) , therefor

$\omega(\Gamma(R))=\sum_{i=1}^{\frac{m-1}{2}}|A_i|+3=p^{\frac{m-1}{2}}+2$.

Theorem2.7:

Let $R=Z_{p^m qr}$ .Then Γ(R) has s-partite, where

$$s=\begin{cases} p^{\frac{m}{2}}+1 & ,\text{if } m \text{ even.} \\ p^{\frac{m-1}{2}}+2 & ,\text{if } m \text{ odd }. \end{cases}$$

Proof:

Let $Z(R)^*=\cup A_i \cup B_j \cup C_k \cup D_l$, where A_i , B_j, C_k , D_l, i= 1 ,2,…,m, j=0,1,2,…,m, k=0,1,…,m, ,l=0,…,m-1.

1. If m is even number, the graph Γ(R) has $p^{\frac{m}{2}}+1$ partitions since into subset with different vertices non-adjacent, as follows: The vertices $\bigcup_{i=1}^{\frac{m}{2}} A_i$ with number $p^{\frac{m}{2}}+1$. we notice that such vertices in only one set . let such set one $x_1, x_2, \ldots x_{p^{\frac{m}{2}}-2}$,$x_{p^{\frac{m}{2}}-1}$, and let $x_{p^{\frac{m}{2}}-1}=\{u : u \in A_{\frac{m}{2}}\}$. Clearly the vertices in $\bigcup_{i=\frac{m}{2}+1}^{m} A_i$,is non-adjacent with each other since i+j > m .and all vertices are non – adjacent with the set $x_{p^{\frac{m}{2}}-1}$. So we can put them in one set namely $U_0=\{x_{p^{\frac{m}{2}}-1}\} \bigcup_{i=\frac{m}{2}+1}^{m} A_i$. We note that the vertices in the set $U_1=\bigcup_{j=0}^{m} B_j$, are non-adjacent vertices with other, and some of its elements is adjacent with x_i , i=1,2,…, $p^{\frac{m}{2}}-1$. In a similar way we can show that the set $U_2=\bigcup_{j=0}^{m} C_j \bigcup_{k=0}^{m-1} D_k$. So that Γ(R) has $p^{\frac{m}{2}}+1$ partitions U_0, U_1, U_2, x_1, $x_2, \ldots, x_{p^{\frac{m}{2}}-2}$ and Γ(R) has $p^{\frac{m}{2}}+1$ partite.

2. If m is odd as in proof 1. we observe that the set $\cup_{i=1}^{\frac{m-1}{2}} A_i$,as adjacent vertices with other. So every vertex of the vertices in an independent set. We also notice that the vertices of the set $\cup_{i=\frac{m+1}{2}}^{m} A_i$, as non- adjacent vertices with other .So we shall put it in one set , namely U_0. In a similar way we can find the element of the sets $U_1 = \cup_{j=0}^{m} B_j$, and also $U_2 = \cup_{j=0}^{m} C_j \cup_{k=0}^{m-1} D_k$, in an independent sets with number s = $p^{\frac{m-1}{2}}+2$ .So the graph has s- partite .

Theorem 2.8:

Let R= $Z_{p^m\,qr}$ .Then $\chi(\Gamma(R))$ =s, where

$$s = \begin{cases} p^{\frac{m}{2}} + 1, & \text{if } m \text{ even}, \\ p^{\frac{m-1}{2}} + 2, & \text{if } m \text{ odd} . \end{cases}$$

Proof:

By Theorem 2.7 Γ(R) has s-partite so that $\chi(\Gamma(R)) \le s$. also Theorem 2.6 shown $\omega(\Gamma(R))$ =s therefore $\chi(\Gamma(R)) \ge s$. So that $\chi(\Gamma(R))$ =s.

3. Hosoya and Wiener index of zero divisor graph of $\Gamma(Z_{p^m qr})$

In this section we shall find the Hosoya polynomials and Wiener index of $\Gamma(Z_{p^m qr})$ we also find the order and size of $\Gamma(Z_{p^m qr})$**,** where $m \ge 2$ and p, q ,r are distinct primes and show diam(Γ (R)) =3.

Lemma 3.1 [11]

Let (R_1, m_1), (R_2, m_2) and (R_3,m_3) are finite local rings, then

$| Z(R_1 \times R_2 \times R_3)^* | = |R_1| . |R_2| . | m_3| + | Z(R_1 \times R_2)| . (|R_3|- | m_3|) -1$, where

$| Z(R_1 \times R_2) | = |R_1| . | m_2| + |R_2| . | m_1| - |m_1|.|m_2|$.

Next we shall give the following results

Theorem 3.2:

Let R = $Z_{p^m qr}$, then Γ(R) has order, $a_0=p^{m-1}(qr+pr-pq+r-q+p-1)-1$ and size, $a_1 = \frac{1}{2}[qr\left(\frac{1-p^m}{1-p}\right) + q\left(\frac{1-p^{m+1}}{1-p}\right) + r\left(\frac{1-p^{m+1}}{1-p}\right) + p\left(\frac{1-p^m}{1-p}\right) - \left(\left\lfloor\frac{m}{2}\right\rfloor + 4m + 2\right)]$.

Proof :

Since $Z_{p^m qr} \cong Z_{p^m} \times Z_q \times Z_r$,then by lemma(3.1) Γ(R) has order

$a_0 = p^{m-1}(qr+pr-pq+r-q+p-1)-1$

Next we shall find size a_1 of R , since

$2a_1 = \sum_{v\in\Gamma(R)} deg(v)$

$= \sum_{v\in A_i} deg(v) + \sum_{v\in B_j} deg(v) + \sum_{v\in C_k} deg(v) + \sum_{v\in D_k} deg(v)$ where i=1,…,m,j=0,…,m,k=0,…,m,l=0,…,m-1

$2a_1 = \sum_{i=1}^{\left\lfloor\frac{m}{2}\right\rfloor}(p^{m-i}\,qr - 2) + \sum_{i=\left\lfloor\frac{m}{2}\right\rfloor+1}^{m}(p^{m-i}\,qr - 1) + \sum_{j=0}^{m}(p^{m-j}\,r - 1) + \sum_{k=0}^{m}(p^{m-k}\,q - 1) + \sum_{l=0}^{m-1}(p^{m-l} - 1)$.

Hence

$a_1 = \frac{1}{2}[qr\left(\frac{1-p^m}{1-p}\right) + q\left(\frac{1-p^{m+1}}{1-p}\right) + r\left(\frac{1-p^{m+1}}{1-p}\right) + p\left(\frac{1-p^m}{1-p}\right) - \left(\left\lfloor\frac{m}{2}\right\rfloor + 4m + 2\right)]$.

Theorem3.3 :

Let R = $Z_{p^m qr}$, then diam $\Gamma(R)$=3.

Proof:

Since $Z(Z_{p^m qr})^*= \cup A_i \cup B_j \cup C_k \cup D_l$, where A_i, B_j, C_k, D_l, i= 1 ,2,…,m, j=0,1,2,…,m, k=0,1,…,m, ,l=0,…,m-1, be subsets as classification 2.2.we can find $v_1 \in B_m$, $v_2 \in C_m$ .Since every element in B_m is adjacent with only element in C_0 and every element in C_m is adjacent with only element in B_0, and since every element in B_0 is adjacent with elements in C_0.Then we have $d(v_1,v_2)$=3. So that diam $\Gamma(Z_{p^m qr})$=3.

Lemma3.4: [9]

Let G be a connected graph of order r, then $\sum_{i=0}^{dima(G)} d(G,i) = \frac{1}{2} r(r+1)$.

In the next result we find the Hosoya polynomials of $\Gamma(Z_{p^n qr})$.

Theorem 3.5 :

Let R= $Z_{Z_{p^m qr}}$, then $H(\Gamma(R),x)=a_0+a_1x+a_2x^2+a_3x^3$, where

$a_0 = p^{m-1} (qr + pr – pq + r – q + p – 1) – 1$,

$a_1 = \frac{1}{2}[qr\left(\frac{1-p^m}{1-p}\right) + q\left(\frac{1-p^{m+1}}{1-p}\right) + r\left(\frac{1-p^{m+1}}{1-p}\right) + p\left(\frac{1-p^m}{1-p}\right) - \left(\left\lfloor\frac{m}{2}\right\rfloor + 4m + 2\right)]$,

$a_2 = \frac{1}{2} (p^{m-1} (qr + pr –pq + r – q + p – 1) - 1) [p^{m-1} (qr + pr –pq + r – q + p – 1)-1+1]- p^{m-1} (qr + pr – pq + r – q + p – 1) – 1 - \frac{1}{2} [qr\left(\frac{1-p^m}{1-p}\right) + q\left(\frac{1-p^{m+1}}{1-p}\right) + r\left(\frac{1-p^{m+1}}{1-p}\right) + p\left(\frac{1-p^m}{1-p}\right) - \left(\left\lfloor\frac{m}{2}\right\rfloor + 4m + 2\right)] - (p^m-p^{m-1}) (q-1)[(p^m-1)+ (p^m-1) (r-1) + p^{m-1} (r-1)^2 + p^{m-1} (q-1)(r-1)+ (p^m -1)(r-1)]$, and

$a_3 = (p^m-p^{m-1}) (q-1)[(p^m-1)+ (p^m-1) (r-1)+p^{m-1} (r-1)^2+ p^{m-1} (q-1)(r-1)+ (p^m -1)(r-1)]$.

Proof:

By Theorem 3.2, $a_0 = p^{m-1} (qr+pr-pq+r-q+p-1) - 1$ and $a_1 = \frac{1}{2}[qr\left(\frac{1-p^m}{1-p}\right) + q\left(\frac{1-p^{m+1}}{1-p}\right) + r\left(\frac{1-p^{m+1}}{1-p}\right) + p\left(\frac{1-p^m}{1-p}\right) - \left(\left\lfloor\frac{m}{2}\right\rfloor + 4m + 2\right)]$. Now to find a_3

let $x, y \in Z^*(Z_{p^m qr}) = \bigcup_{i=1}^{m} A_i \bigcup_{j=1}^{m} B_j \bigcup_{k=1}^{m} C_k \bigcup_{l=1}^{m-1} D_l$,

where A_i , B_j, C_k, D_l as classification 2.2.Then there are four cases:

Case1:

1. If $x \in A_i$ and $y \in A_j$, i=1,2,…,m,j=1,2,…,m.$x=k_1p^{m-i} qr$,$p^{m-i+1} \nmid k_1$, $y=k_2p^{m-j} qr$, $p^{m-j+1} \nmid k_2$, then x and y adjacent $p^{m-1}qr$,that is $d(x,y) \leq 2$ a contradiction .
2. If $x \in A_i$ and $y \in B_j$, i=1,2,…,m,j=0,1,2, …,m.$x=k_1p^{m-i} qr$,$p^{m-i+1} \nmid k_1$, $y=k_2p^{m-j} q$, $p^{m-j+1} \nmid k_2$,then x and y adjacent $p^{m-1}q r$,that is $d(x,y) \leq 2$ a contradiction.
3. If $x \in A_i$ and $y \in D_l$, i=1,2,…,m, l=0,1,2,…,m-1. $x=k_1p^{m-i} qr$,$p^{m-i+1} \nmid k_1$, $y=k_2p^{m-l}$, $p^{m-l+1} \nmid k_2$, then x and y adjacent $p^{m-1}qr$, that is $d(x,y) \leq 2$ a contradiction
4. If i=m ,$x \in A_i$, then x adjacent with only elements in B_0 or C_0 , so that d(x,y)=3 for any k=1,2,…,m-1 and the number of elements in this case is $|A_m|.\sum_{k=1}^{m-1} |C_k|=(p^m-p^{m-1}).(p^m-1)(q-1)$.

Case2:

1 If $x \in B_i$ and $y \in B_j$, i, j=0,1,2,…,m. $x=k_1p^{m-i} q$, $y=k_2p^{m-j} q$, $p^{m-i+1}q \nmid k_1$,$p^{m-j+1}q \nmid k_2$,then x and y adjacent $p^{m-1}q$,that is $d(x,y) \leq 2$ a contradiction .

2. If $x\in B_i$ and $x\in C_k$, i, j=0,1,2,…,m. $x=k_1p^{m-i}q$, $y=k_2p^{m-k}r$, $p^{m-i+1}\nmid k_1$ $p^{m-i+1}\nmid k_1$, $p^{m-k+1}r\nmid k_2$,then x and y adjacent $p^{m-1}q$, that is $d(x,y)\leq 2$ a contradiction .
3. If $x\in B_i$ and $y\in D_l$, i=0,1,2,…,m ,l=0,1,2,…,m-1,$x=k_1p^{m-i}q$, $y=k_2p^{m-l}$, $p^{m-i+1}q\nmid k_1$,$p^{m-l+1}q\nmid k_2$,then x and y adjacent $p^{m-1}q$, that is $d(x,y)\leq 2$ a contradiction.
4. If i=m , $x\in B_i$, and $x\in C_k$ then x adjacent with only elements in C_0 , so that d(x,y)=3 for any k=1,2,…,m-1 and the number of element in this cases $|B_m|.\sum_{k=1}^{m-1}|C_k|=(p^m-p^{m-1})(r-1).(p^{m-1}-1)(q-1)$.
5. If i=m , $x\in B_i$ and $y\in D_l$ then x adjacent with only elements in C_0 , so that d(x,y)=3 for any l=0,1,2,…,m-1 and the number of element in this cases $|B_m|.\sum_{l=0}^{m-1}|D_l|=(p^m-p^{m-1})(r-1).\ p^{m-1}(q-1)(r-1)$.

Case3:

1. If $x\in C_i$ and $x\in C_k$,i , j=0,1,2,…,m. $x=k_1p^{m-i}r$, $y=k_2p^{m-k}r$, $p^{m-i+1}r\nmid k_1$,$p^{m-k+1}r\nmid k_2$,then x and y are adjacent $p^{m-1}r$, that is $d(x,y)\leq 2$ a contradiction .
2. If $x\in C_i$, and $y\in D_l$, i ,j=0,1,2,…,m-1. $x=k_1p^{m-i}r$, $y=k_2p^{m-l}$, $p^{m-i+1}r\nmid k_1$,$p^{m-k+1}\nmid k_2$,then x and y adjacent $p^{m-1}r$, that is $d(x,y)\leq 2$ a contradiction.
3. If i=m ,$x\in C_i$ and $y\in B_j$ then x adjacent with only elements in B_0 , so that d(x,y)=3 for any j=1,2,…,m and the number of element in this cases $|C_m|.\sum_{j=1}^{m}|B_j|=(p^m-p^{m-1})(q-1).\ p^{m-1}(r-1)$
4. If i=m , $x\in C_i$,and $y\in D_l$ then x adjacent with only elements in B_0 , so that d(x,y)=3 for any l=0,1,2,…,m and the number of element in this cases $|C_m|.\sum_{l=1}^{m-1}|D_l|=(p^m-p^{m-1})(q-1).(p^{m-1}-1)(r-1)(q-1)$

Case4:

If $x\in D_i$ and $y\in D_l$, i,l=0,1,2,…,m-1. $x=k_1p^i$, $y=k_2p^l$, $q\nmid k_1$,$q\nmid k_2$,then x and y adjacent qr, that is $d(x,y)\leq 2$ a contradiction.

Therefore, $a_3=(p^m-p^{m-1})(q-1)[(p^m-1)+(p^m-1)(r-1)+p^{m-1}(r-1)^2+p^{m-1}(q-1)(r-1)+(p^m-1)(r-1)]$.

Finally we find a_2 by using lemma 3.4. , then

$a_2=\frac{1}{2}(p^{m-1}(qr+pr-pq+r-q+p-1)-1)[p^{m-1}(qr+pr-pq+r-q+p-1)-1+1]-p^{m-1}(qr+pr-pq+r-q+p-1)-1-\frac{1}{2}[qr\left(\frac{1-p^m}{1-p}\right)+q\left(\frac{1-p^{m+1}}{1-p}\right)+r\left(\frac{1-p^{m+1}}{1-p}\right)+p\left(\frac{1-p^m}{1-p}\right)-\left(\left\lfloor\frac{m}{2}\right\rfloor+4m+2\right)]-(p^m-p^{m-1})(q-1)[(p^m-1)+(p^m-1)(r-1)+p^{m-1}(r-1)^2+p^{m-1}(q-1)(r-1)+(p^m-1)(r-1)]$.

Corollary 3.6:

$W(\Gamma(Z_{p^m qr}))=\frac{1}{2}[qr\left(\frac{1-p^m}{1-p}\right)+q\left(\frac{1-p^{m+1}}{1-p}\right)+r\left(\frac{1-p^{m+1}}{1-p}\right)+p\left(\frac{1-p^m}{1-p}\right)-\left(\left\lfloor\frac{m}{2}\right\rfloor+4m+2\right)]$ $+2[\frac{1}{2}(p^{m-1}(qr+pr-pq+r-q+p-1)-1)[p^{m-1}(qr+pr-pq+r-q+p-1)-1+1]-p^{m-1}(qr+pr-pq+r-q+p-1)-1-\frac{1}{2}[qr\left(\frac{1-p^m}{1-p}\right)+q\left(\frac{1-p^{m+1}}{1-p}\right)+r\left(\frac{1-p^{m+1}}{1-p}\right)+p\left(\frac{1-p^m}{1-p}\right)-\left(\left\lfloor\frac{m}{2}\right\rfloor+4m+2\right)]-(p^m-p^{m-1})(q-1)[(p^m-1)+(p^m-1)(r-1)+p^{m-1}(r-1)^2+p^{m-1}(q-1)(r-1)+(p^m-1)(r-1)]]$ $+3[(p^m-p^{m-1}).(p^m-1)(q-1)+(p^m-p^{m-1})(r-1).(p^m-1)(q-1)+(p^m-p^{m-1})(r-1).\ p^{m-1}(q-1)(r-1)+(p^m-p^{m-1})(q-1).\ p^{m-1}(q-1)(r-1)+(p^m-p^{m-1})(q-1).(p^m-1)(r-1)]$.

Proof: Since $W(\Gamma(Z_{p^m qr}))=\frac{d}{dx}H(\Gamma(Z_{p^m qr});x)|_{x=1}$ therefore,

$W(\Gamma(Z_{p^m qr}))=0+a_1+2a_2+3a_3=\frac{1}{2}[pq\left(\frac{1-p^m}{1-p}\right)+q\left(\frac{1-p^{m+1}}{1-p}\right)+r\left(\frac{P-p^{m+1}}{1-p}\right)+p\left(\frac{1-p^m}{1-p}\right)-\left(\left\lfloor\frac{m}{2}\right\rfloor+4m+2\right)]$,+2 $[\frac{1}{2}\ (p^{m-1}(qr+pr-pq+r-q+p-1)-1)\ [p^{m-1}(qr+pr-pq+r-q+p-1)-1+1]-p^{m-1}(qr+pr-pq+r-q+p-1)-1-\frac{1}{2}\ [qr\left(\frac{1-p^m}{1-p}\right)+q\left(\frac{1-p^{m+1}}{1-p}\right)+r\left(\frac{1-p^{m+1}}{1-p}\right)+p\left(\frac{1-p^m}{1-p}\right)-\left(\left\lfloor\frac{m}{2}\right\rfloor+4m+2\right)]-(p^m-p^{m-1})(q-1)[(p^m-1)+(p^m-1)(r-1)+p^{m-1}(r-1)^2+p^{m-1}(q-1)(r-1)+(p^m-1)(r-1)]]+3[(p^m-p^{m-1}).(p^m-1)(q-1)+(p^m-p^{m-1})(r-1).(p^m-1)(q-1)+(p^m-p^{m-1})(r-1).\ p^{m-1}(q-1)(r-1)+(p^m-p^{m-1})(q-1).\ p^{m-1}(q-1)(r-1)+(p^m-p^{m-1})(q-1).(p^m-1)(r-1)]$.

Example 2:

In Example 1,we find Hosoya polynomial and Wiener index of zero divisor graph of $\Gamma(Z_{2^5.3.5})$,$d(\Gamma(Z_{2^5.3.5},0))=351$, $d(\Gamma(Z_{2^5.3.5},1))=504$, $d(\Gamma(Z_{2^5.3.5},2))=39705$, $d(\Gamma(Z_{2^5.3.5},3))=21216$.

Therefore we have $W(\Gamma(Z_{2^5.3.5});x)|_{x=1}=351+504x+39705x^2+21216x^3$. By definition Wiener index of zero divisor graph of $\Gamma(Z_{2^5.3.5})$.

$W(\Gamma(Z_{2^5.3.5}))=504+2(39705)+3(21216)=143562$.

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