The Numerical Range of 6 × 6 Irreducible Matrices

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ABASTRACT

In this paper, we consider the problem of characterizing the numerical range of 6 by 6 irreducible matrices which have line segments on their boundary.

Keywords: numerical range, irreducible matrices.

المدى العددي للمصفوفات اللااختزالية من الرتبة *6 × 6*

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الملخص

في هذا البحث تمت دراسة مسألة خواص المدى العددي للمصفوفات اللااختزالية من الرتبة 6x6التي لها مستقيمات مقطعية على تخومها.

الكلمات المفتاحية: المدى العددي، المصفوفات اللااختزالية.

1. Introduction

Let M_n (C) be the algebra of n × n complex matrices. The numerical range of $A \in M_n$ is defined by $W(A)=\{x^*Ax: x \in C^n, x^*x=1\}$[4], where x^* the adjoint of $\boldsymbol{x} \in$ C is defined by $\boldsymbol{x}^*=\bar{x}^T$ where $\bar{x}$ is the component-wise conjugate, and $\boldsymbol{x}^{\mathbf{T}}$ is the transpose of x [4]. As pointed out by many authors, for2×2 matrices A a complete description of the numerical range W(A) is well-Known. Namely, W(A) is an ellipse with foci at the eigenvalues λ_1, λ_2 of A and a minor axis of the length s =(trace(A*A)- $|\lambda_1|^2 - |\lambda_2|^2)^{\frac{1}{2}}$. In [4], of course, s=0 for normal A, and the ellipse in this case degenerates into a line segments connecting $\lambda_1$with λ_2 . On the other hand ,for 2×2 matrices A with coinciding eigenvalues the ellipse W(A) degenerates into a disk. For3×3 matrix A, this was first done by Kippenhahn. In[6] , his characterization is based on the factorability of the associated polynomial $P_A(x,y,z)=\det(xReA+yImA+zI_3)$ .This was improved in[5] by expressing the condition in terms of entries of A, also for 4×4 and 5×5 matrices A, this was improved in [2] by expressing the conditions in terms of entries of A. The aim of this paper is to give a sufficient and necessary condition for numerical range of 6 by 6 matrix with a line segment on its boundary .

2. Preliminaries

In the following, we give some definitions and results on W(A) that are useful in this study.

Definition 2.1 [4] A matrix $A \in M_n(C)$ is said to be irreducible if either n=1 or $n \geq 2$ and there does not exist a permutation matrix $P \in M_n(C)$ such that $P^T AP = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$ where B, D are nonempty square matrices.

Definition 2.2 [4] A matrix $B \in M_n(c)$ such that $x^*Bx \geq 0$ for all $x \in C^n$, is said to be positive semidefinite .

Proposition 2.3 [4] The numerical range of $A \in M_n$ is always a compact convex set in C. It contains the spectrum $\sigma(A)$ of A and is equal to the convex hull of $\sigma(A)$ if A is normal.

Proposition 2.4[4] Let $A \in M_n(C)$

(a) $W(A) = W(A^t)$
(b) W(A) = W(U*AU) for any unitary U.
(c) $W(\lambda A)= \lambda W(A)$ for any $\lambda \in C$.
(d) $W(\lambda I+A) = \lambda+W(A)$ for any $\lambda \in C$.
(e) $W(A^*) = \{z: z \in W(A).\}$
(f) W(ReA)=ReW(A) and W(ImA)=ImW(A). Here ReA= (A*+A)/2 and ImA= (A*-A)/2i are the real and imaginary part of A respectively.

Proposition 2.5 [4] Let $A \in M_n(C)$ and *a,b* be scalars

(a) $W(A) =\{\lambda\}$ if and only if $A=\lambda I$.
(b) W(A) is contained in *a* straight line of the plane if and only if A=*a*B+*b*I for some Hermitian matrix B.In particular in this case A is normal.
(c) $W(A)\geq 0$ if and only if A is positive semidefinite.

Proposition 2.6 [4] Suppose that B is a principal submatrix of $A \in M_n(C)$. Then $W(B) \subseteq W(A)$.

We now relate the numerical range of an $n \times n$ matrix to an algebraic curve of class n. The next proposition indicates how the characteristic polynomial of some pencil associated with the matrix arises in this connection.

Proposition 2.7 [7] Let $A \in M_n(C)$. If ax+by +c=0 is a supporting line of W(A), then $\det(aReA+bImA+cI_n)=0$

It follows from the above proposition that when studying the numerical range of A it is sensible to consider the algebric curve C(A) which is dual to the one given by $P_A(xReA+yImA+zI_n)=0$.

Note that $P_A(-x,-y,z)=\det(zI_n-xReA-yImA)$ is nothing, but the characteristic polynomial of the pencil xReA+yImA. It is easily to see that P_A is a homogenous polynomial of degree n with real coefficients. Thus, in particular, the curve C(A) is of class n and has n real eigenvalues of A.

Proposition 2.8 [7] If the eigenvalues of the $n \times n$ matrix A are a_j+ib_j , j=1,…,n where the a_j and b_j are real, then the real foci of the algebraic curve C(A) are exactly the points (a_j,b_j) , j=1,…,n.

Note that the proposition (2.7 together with the duality, implied that any supporting line of W(A) is tangent to C(A). The following proposition gives a more precise relation between W(A) and C(A).

Proposition 2.9 [6] If A is a $n \times n$ matrix, then its numerical range W(A) is the convex hull of the real points of the curve C(A).
The real part of the curve C(A) in the complex plane namely the set$\{a+ib \in C : a,b \in R$ and $ax+by+z=0$ is tangent to $P_A(x,y,z)=0\}$, will be denoted by $C_R(A)$ and is called the Kippenhahn curve of A.

3. Line segments of the Boundary of Numerical Range

In the following we will restrict ourselves to the irreducible matrix The next theorem gives conditions for the numerical range to have a line segments on its boundary .

Theorem 3.1 Let A be an irreducible matrix. Then the following statements are equivalent:

- **(a)** W(A) has a line segments on its boundary.
- **(b)** $0< \text{rank}(uReA+vImA+wI_6) \leq 4$ for some real numbers u,v and w.(u and v not both zero).
- **(c)** uReA+vImA has a multiple eigenvalue for some real u and v which are not both zero.

Proof: W(A) has a line segments on the boundary of $C_R(A)$ has a double or triple or quadripartite or tangent $ux+vy+w=0$ (u and v not both zero) . This corresponds to a root *w* of the equation $\det(uReA+vImA+zI_6)=0$ with multiplicity 2 or 3 or 4 or 5, which is the same as saying that $uReA+vImA+zI_6$ has rank 4 or 3 or 2 or 1. This proves the equivalence of (a) and (b).

(b)$\Rightarrow$ (c) : Suppose (b) holds then –*w* is an eigenvalue of uReA+vImA with multiplicity 5, and we get double or triple or quadripartite or tangent .Thus (b) implies (c). To prove (c)$\Rightarrow$ (b), assume that –w is a multiple eigenvalue of uReA+vImA.If –w is of multiplicity 6 , then the Hermition uReA+vImA would be a scalar matrix. In this case, ReA and ImA would commute and hence A would be normal, contradicting the irreduciblity of A. Thus (c) impies (b).

Corollary 3.2 Le A be an irreducible matrix and unitary similar to real matrix. Then W(A) has a line segments on its boundary if and only if ReA has a multiple eigenvalue.

Proof: If A is a real matrix. Then W(A) is symmetric about the real axis. Hence the line segment of the boundary of W(A) must be a vertical line, by theorem(3.1) the matrix 1.ReA+0.ImA=ReA has a multiple eigenvalue.

We begin by deriving a canonical form for an irreducible form for an irreducible 6 $\times$ 6 matrix with a line segment on the boundary of it is numerical range, if W(A) has a line segment on its boundary . After rotation,Shifting, and multiplication by a positive number, we may assume that a line segment stretches from 0 to i. Since W(A) is convex, it must be contained entirely in the right or left half-plane. Applying yet another rotation and translation, if necessary we may assume that W(A) is in the right half-plane. By theorem(3.1) we have rankA=1 or 2 or 3. Therefore we will discus these two cases, respectively.

Theorem 3.3 Let A be an irreducible matrix. Then W(A) has a line segment extending from 0 to i on its boundary and rankA=1 if and only if A may be written in the form A=

$$\begin{bmatrix} i & 0 & 0 & 0 & 0 & -c_1 \\ 0 & 0 & 0 & 0 & 0 & -c_2 \\ 0 & 0 & 0 & 0 & e_1 & -e_2 \\ 0 & 0 & 0 & e_1 & e_2 & -e_3 \\ 0 & 0 & e_1 & e_2 & e_3 & -e_4 \\ c_1 & c_2 & e_2 & e_3 & e_4 & -e_6 \end{bmatrix}$$

,where $c_1,c_2,e_2,e_3,e_4,$and real part of e_6 are positive and $e_1,e_2, e_3 \in[0,i]$.

Proof: Since W(A) is contained in the closed right half –plane, ReA is positive semi definite. By assumption, kerReA is 5-dimensional subspace, we may represent the line transformation of A restricted to kerReA, $A_{|\ker \operatorname{Re} A} = A'$ by a 5 $\times$ 5 matrix. By choosing a proper basis for A, A' is the leading principal submatrix of A. since 0 and i are in W(A), there exists unit vectors $x_1,x_2 \in C^6$ such that $<Ax_1,x_1>=i$ and $<Ax_2,x_2>=0$. It is clear that $x_1,x_2 \in \ker \mathrm{Re}A$, since $\mathrm{Re}A \geq 0$ and $<(\mathrm{Re}A)x_j,x_j>=0$, j=1,2. It follows that the line segment [0, i],is contained in W(A'). Also W(A')$\subseteq$ W(A) . since $A_{|\ker \mathrm{Re} A} = A'$, we have Re A'=0 and ReW(A')=W(Re A')={0}. We thus get W(A')=[0,i]. This implies that A' is normal with eigenvalues 0 and i so

with proper basis $A' = \begin{bmatrix} i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_1 \\ 0 & 0 & 0 & e_1 & e_2 \\ 0 & 0 & e_1 & e_2 & e_3 \end{bmatrix}$ and $A = \begin{bmatrix} i & 0 & 0 & 0 & 0 & v_1 \\ 0 & 0 & 0 & 0 & 0 & v_2 \\ 0 & 0 & 0 & 0_1 & e_1 & e_2 \\ 0 & 0 & 0 & e_1 & e_2 & e_3 \\ 0 & 0 & e_1 & e_2 & e_3 & e_4 \\ c_1 & c_2 & e_2 & e_3 & e_4 & e_6 \end{bmatrix}$

where $e_1,e_2, e_3 \in [0,i]$ are pure imaginary.

Since ReA is positive semi definite, a calculation shows that:

$$2\mathrm{Re}A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & v_1+\bar{c}_1 \\ 0 & 0 & 0 & 0 & 0 & v_2+\bar{c}_2 \\ 0 & 0 & 0 & 0 & e_1+\bar{e}_2 & e_2+\bar{e}_3 \\ 0 & 0 & 0 & e_1+\bar{e}_2 & e_2+\bar{e}_3 & e_3+\bar{e}_4 \\ 0 & 0 & e_1+\bar{e}_2 & e_2+\bar{e}_3 & e_3+\bar{e}_4 & e_4+\bar{e}_5 \\ c_1+\bar{v}_1 & c_2+\bar{v}_2 & e_2+\bar{e}_3 & e_3+\bar{e}_4 & e_4+\bar{e}_5 & 2\mathrm{Re}\, e_6 \end{bmatrix},$$

And therefore $\mathrm{Re}e_6 \geq 0, v_1 = -\bar{c}_1, v_2 = -\bar{c}_2, e_2 = -\bar{e}_3, e_3 = -\bar{e}_4 \; and \; e_4 = -\bar{e}_5$ Moreover, $\mathrm{Re}e_6$ must be positive, because rankReA=1.By a diagonal unitary similarity, we may assume that c_1,c_2,e_2,e_3, e_4 are non-negative. If one of them is 0, then A is reducible, so c_1,c_2,e_2,e_3, e_4 are positive. Now suppose that A is in the form expressed in the theorem. Consider the principal submatrix A' from the first five rows and columns of A, W(A') is a line segment from 0 to i . clearly, W(A')$\subseteq$ W(A). But since ReA is positive semidefinite, W(A) lies entirely in the right half-plane. So the line segment from 0 to i must be on the boundary of W(A). To see that the line segment does not go beyond o or i, note that any point α on that line must be pure imaginary. So if

$\alpha=<Ax,x>=<(\mathrm{Re}A)x,x>+i<(\mathrm{Im}A)x,x>$, then $<(\mathrm{Re}A)x,x>=0$.
Hence

$x \in \ker \mathrm{Re}A = \mathrm{span}\{[1,0,0,0,0,0]^T,[0,1,0,0,0,0]^T,[0,0,1,0,0,0]^T,[0,0,0,1,0,0]^T,[0,0,0,0,1,0]^T\}$ if $\|x\| = 1$, then $x=[v_1,v_2,v_3,v_4,v_5,0]^T$

with $|v_1|^2 + |v_2|^2 + |v_3|^2 + |v_4|^2 + |v_5|^2 = 1$, and

$0 \le \langle (\mathrm{Im}\, A)x, x\rangle = |v_1|^2 + |e_1||v_3|^2 + |e_2||v_4|^2 + |e_5||v_5|^2 \le$

$|v_1|^2 + |v_3|^2 + |v_4|^2 + |v_5|^2 \le 1.$

Theorem 3.4 Let A be an irreducible matrix written in the form

$$A=\begin{bmatrix} h_1 & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ \bar{h}_{12} & h_2 & h_{23} & h_{24} & h_{25} & h_{26} \\ \bar{h}_{13} & \bar{h}_{23} & h_3 & h_{34} & h_{35} & h_{36} \\ \bar{h}_{14} & \bar{h}_{24} & \bar{h}_{34} & h_4 & h_{45} & h_{46} \\ \bar{h}_{15} & \bar{h}_{25} & \bar{h}_{35} & \bar{h}_{45} & h_5 & h_{56} \\ \bar{h}_{16} & \bar{h}_{26} & \bar{h}_{36} & \bar{h}_{46} & \bar{h}_{56} & h_6 \end{bmatrix} + i \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_6 \end{bmatrix}$$

where $k_1, k_2, k_3, k_4, k_5, k_6$ *are distinct*. Then W(A) has a line segment on its boundary if and only if

$$h6(k4 - k5) + h_5(k6 - k4) + h4(k5 - k6) = (k4 - k5)\left(\frac{h_{45}\bar{h}_{46}}{h_{45}}\right) + (k_6 - k4)\left(\frac{h_{35}\bar{h}_{45}}{h_{36}}\right) + (k5 - k6)\left(\frac{h_{36}\bar{h}_{56}}{h_{35}}\right) \quad (3.1)$$

$$h_6(k3-k4)+h_4(k6-k3)+h_3(k4-k6)=(k3-k4)\left(\frac{h_{36}\bar{h}_{34}}{h_{46}}\right)+(k6-k3)\left(\frac{h_{24}\bar{h}_{36}}{h_{23}}\right)+(k4-k6)\left(\frac{h_{46}\bar{h}_{46}}{h_{45}}\right) \quad (3.2)$$

$$h_6(k2-k3)+h3(k6-k2)+h2(k3-k6)=(k2-k3)\left(\frac{h_{26}\bar{h}_{23}}{h56}\right)+(k6-k2)\left(\frac{h_{24}\bar{h}_{26}}{h34}\right)+(k3-k6)\left(\frac{h_{23}\bar{h}_{26}}{h_{23}}\right) \quad (3.3)$$

$$h_1(k5-k_6)+h_5(k_6-k_1)+h6(k_1-k5)=(k5-k_6)\left(\frac{h_{35}\bar{h}_{46}}{h_{35}}\right)+(k_6-k_1)\left(\frac{h_{16}\bar{h}_{15}}{h_{56}}\right)+(k_1-k_5)\left(\frac{h_{56}\bar{h}_{56}}{h_{26}}\right) \quad (3.4)$$

$h_{12}h_{26}\bar{h}_{16}$, $h_{13}h_{36}\bar{h}_{16}$, $h_{14}h_{46}\bar{h}_{16}$, $h_{15}h_{56}\bar{h}_{16}$, $h_{35}h_{56}\bar{h}_{36}$, $h_{34}h_{45}\bar{h}_{36}$, $h_{24}h_{35}\bar{h}_{26}$, $h_{23}h_{34}\bar{h}_{26}$,

$h_{25}h_{36}\bar{h}_{26}$, $h_{45}h_{56}\bar{h}_{46}$ are real with $h_{12}h_{13}h_{14}h_{15}h_{16}h_{23}h_{24}h_{25}h_{26}h_{34}h_{35}h_{36}h_{45}h_{46}h_{56} \ne 0$. (3.5)

Proof: suppose rank B=rank(u ReA+v Im A+w I_6)=1, since the eigenvalues of Im A are all distinct, it is possible only when u is non zero. With-out loss of generality we may assume that u=1. To simplify further calculations, rewrite B in the form

$$B=\begin{bmatrix} h_1'+vk_1'+w' & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ \bar{h}_{12} & \bar{h}_2+vk_2'+w' & h_{23} & h_{24} & h_{25} & h_{26} \\ \bar{h}_{13} & \bar{h}_{23} & \bar{h}_3+vk_3'+w' & h_{34} & h_{35} & h_{36} \\ \bar{h}_{14} & \bar{h}_{24} & \bar{h}_{34} & \bar{h}_4+vk_4'+w' & h_{45} & h_{46} \\ \bar{h}_{15} & \bar{h}_{25} & \bar{h}_{35} & \bar{h}_{45} & \bar{h}_5+vk_5'+w' & h_{56} \\ \bar{h}_{16} & \bar{h}_{26} & \bar{h}_{36} & \bar{h}_{46} & \bar{h}_{56} & w' \end{bmatrix}$$

Where $w' = w + h6 + vk_6$, $h_i' = h_i - h6$, $k_i' = k_i - k_6$ (i=1, 2, 3, 4,5*)*, since by assumption rank(B)=1, then
$h_{12}h_{13}h_{14}h_{15}h_{16}h_{23}h_{24}h_{25}h_{26}h_{34}h_{35}h_{36}h_{45}h_{46}h_{56} \neq 0$ and

$$\frac{w'}{\bar{h}_{56}} = \frac{h_{56}}{h_5'+vk_5'+w'} = \frac{h_{16}}{h_{15}} = \frac{h_{26}}{h_{25}} = \frac{h_{36}}{h_{35}} = \frac{h_{46}}{h_{45}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.6)$$

$$\frac{w'}{\bar{h}_{46}} = \frac{h_{56}}{\bar{h}_{45}} = \frac{h_{46}}{h_4'+vk_4'+w'} = \frac{h_{36}}{h_{34}} = \frac{h_{26}}{h_{24}} = \frac{h_{16}}{h_{14}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.7)$$

$$\frac{w'}{\bar{h}_{36}} = \frac{h_{56}}{h'_{35}} = \frac{h_{46}}{h'_{34}} = \frac{h_{36}}{h_3'+vk_3'+w'} = \frac{h_{26}}{h_{23}} = \frac{h_{16}}{h_{13}} \quad \ldots\ldots\ldots\ldots\ldots \quad (3.8)$$

$$\frac{w'}{\bar{h}_{26}} = \frac{h_{56}}{h'_{25}} = \frac{h_{46}}{h'_{24}} = \frac{h_{36}}{h'_{23}} = \frac{h_{26}}{h'_2+vk'_2+w'} = \frac{h_{16}}{h_{12}} \quad \ldots\ldots\ldots\ldots\ldots \quad (3.9)$$

$$\frac{w'}{\bar{h}_{16}} = \frac{h_{56}}{h'_{16}} = \frac{h_{46}}{h'_{14}} = \frac{h_{36}}{h'_{13}} = \frac{h_{26}}{h'_{12}} = \frac{h_{16}}{h'_1+vk'_1+w'} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.10)$$

Solving (3.6), (3.7), (3.8), (3.9) and (3.10) with respect to v , w' we find that

$$w' = \frac{\bar{h}_{56}h_{46}}{h_{45}} = \frac{\bar{h}_{56}h_{36}}{h_{35}} = \frac{\bar{h}_{56}h_{26}}{h_{25}} = \frac{h_{56}\bar{h}_{16}}{h_{15}} = \frac{h_{16}\bar{h}_{46}}{h_{14}} = \frac{h_{26}\bar{h}_{46}}{h_{24}} = \frac{\bar{h}_{46}h_{36}}{h'_{34}} = \frac{\bar{h}_{46}h_{56}}{\bar{h}_{45}}$$

$$= \frac{\bar{h}_{36}h_{16}}{h_{13}} = \frac{\bar{h}_{36}h_{26}}{\bar{h}_{23}} = \frac{h_{36}\bar{h}_{46}}{\bar{h}_{34}} = \frac{\bar{h}_{36}h_{56}}{\bar{h}_{35}} = \frac{\bar{h}_{25}h_{16}}{\bar{h}_{12}} = \frac{h_{26}h_{36}}{h'_{23}} = \frac{\bar{h}_{26}h_{46}}{h'_{24}} = \frac{\bar{h}_{26}h_{56}}{h'_{25}}$$

$$= \frac{h'_{16}h_{26}}{h'_{12}} = \frac{h'_{16}h_{36}}{h'_{13}} = \frac{\bar{h}_{16}h_{46}}{h'_{14}} = \frac{\bar{h}_{16}h_{56}}{h'_{15}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.11)$$

and

$$v=\frac{1}{k'_1}(\frac{\bar{h}_{15}h_{16}}{h_{56}}-h'_1-w')=\frac{1}{k'_1}(\frac{h_{16}\bar{h}_{14}}{h_{46}}-h'_1-w')=\frac{1}{\bar{k}_1}(\frac{h_{16}\bar{h}_{13}}{h_{36}}-h'_1-w')=\frac{1}{\bar{k}_1}(\frac{h_{16}\bar{h}_{12}}{h_{26}}-h'_1-w')$$

$$=\frac{1}{k'_2}(\frac{\bar{h}_{25}h_{26}}{h_{56}}-h'_2-w')=\frac{1}{k'_2}(\frac{\bar{h}_{24}h_{26}}{h_{46}}-h'_2-w')=\frac{1}{k'_2}(\frac{\bar{h}_{23}h_{26}}{h_{36}}-h'_2-w')=\frac{1}{k'_2}(\frac{h_{16}h_{26}}{h_{16}}-h'_2-w')$$

$$=\frac{1}{k'_3}(\frac{\bar{h}_{35}h_{36}}{h_{56}}-h'_3-w')=\frac{1}{k'_3}(\frac{\bar{h}_{34}h_{36}}{h_{46}}-h'_3-w')=\frac{1}{k'_3}(\frac{h_{23}h_{26}}{h_{26}}-h'_3-w')=\frac{1}{k'_3}(\frac{h_{36}h_{13}}{h_{16}}-h'_3-w')$$

$$=\frac{1}{k'_4}(\frac{\bar{h}_{45}h_{46}}{h_{56}}-h'_4-w')=\frac{1}{k'_4}(\frac{h_{34}h_{46}}{h_{36}}-h'_4-w')=\frac{1}{k'_4}(\frac{h_{46}h_{24}}{h_{26}}-h'_4-w')=\frac{1}{k'_4}(\frac{h_{46}h_{14}}{h_{16}}-h'_4-w')$$

$$=\frac{1}{k'_5}(\frac{h_{45}h56}{h_{46}}-h'_5-w')=\frac{1}{k'_5}(\frac{h_{56}h_{36}}{h_{36}}-h'_5-w')=\frac{1}{k'_5}(\frac{h_{56}h_{25}}{h_{26}}-h'_5-w')=\frac{1}{k'_5}(\frac{h_{56}h_{15}}{h_{16}}-h'_5-w') \quad ......(3.12)$$

For convenience, let $w'=\frac{\bar{h}_{56}h46}{h_{45}}$ in the remaining proof, since $h_{12}h_{13}h_{14}h_{15}h_{16}h_{23}h_{24}h_{25}h_{26}h_{34}h_{35}h_{36}h_{45}h_{46}h_{56}\neq 0$ and w' is real, the equality (3.11)yields, (3.5) and (3.6) from (3.12) we have

$$k'_4(\frac{h_{45}h_{56}}{h_{46}}-\frac{\bar{h}_{56}h_{46}}{h_{45}}-h'_5)=k'_5(\frac{\bar{h}_{45}h_{46}}{h_{56}}-\frac{\bar{h}_{56}h_{46}}{h_{45}}-h'_4) \quad ...(3.13)$$

$$k'_3(\frac{\bar{h}_{45}h_{46}}{h_{56}}-\frac{\bar{h}_{56}h_{46}}{h_{45}}-h'_4)=k'_4(\frac{\bar{h}_{35}h_{36}}{h_{56}}-\frac{\bar{h}_{56}h_{46}}{h_{45}}-h'_3) \quad ...(3.14)$$

$$k'_2(\frac{\bar{h}_{35}h_{36}}{h_{56}}-\frac{\bar{h}_{56}h_{46}}{h_{45}}-h'_3)=k'_3(\frac{\bar{h}_{25}h_{26}}{h_{56}}-\frac{\bar{h}_{56}h_{46}}{h_{45}}-h'_2) \quad ...(3.15)$$

$$k'_1(\frac{h_{56}h_{35}}{h_{36}}-\frac{\bar{h}_{56}h_{46}}{h_{45}}-h'_5)=k'_4(\frac{\bar{h}_{15}h_{16}}{h_{56}}-\frac{\bar{h}_{56}h_{46}}{h_{45}}-h'_1) \quad ...(3.16)$$

It is easy to check that (3.1),(3.2), (3.3), (3.4) and (3.5) follow from (3.13),(3.14) and (3.15) and (3.16).

To prove the converse assume that (3.1),(3.2), (3.3), (3.4), (3.5) hold, $w'=\frac{\bar{h}_{56}h_{46}}{h_{45}}$ by (3.1),(3.2), (3.3), (3.4) and (3.5) we obtain (3.13),(3.14) ,(3.15) and (3.16) where $h'_i=h_i-h_6$, $k'_i=k_i-k_6$ *i=(1,2,3,4,5)*,moreover we set

$$v = \frac{1}{k'_1}(\frac{\bar{h}_{15}\, h_{16}}{h_{56}} - w' - h'_1) = \frac{1}{k'_2}(\frac{\bar{h}_{25}\, h_{26}}{h_{56}} - w' - h'_2) = \frac{1}{k'_3}(\frac{\bar{h}_{35}\, h_{36}}{h_{56}} - w' - h'_3)$$

$$= \frac{1}{k'_4}(\frac{\bar{h}_{45}\, h_{46}}{h_{56}} - w' - h'_4) = \frac{1}{k'_5}(\frac{h_{45}\, h_{56}}{h_{46}} - w' - h'_5) \quad \ldots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.17)$$

By taking $w' = \frac{\bar{h}_{56}\, h_{46}}{h_{45}}$ and (3.17) , we conclude that (3.6),(3.7),(3.8),(3.9) and (3.10) hold.

Finally, choose $u = 1$, $w = w' - h_6 - vk6$, then u ReA+v Im A+w I_6 is in the form (M) and has rank 1, hence by theorem (3.1) W(A) has a line segment on its boundary.

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