## On Rings Whose Simple Singular R-Modules Are Flat, I

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**ABSTRACT**

In this paper we investigate von Neumann regularity of rings whose simple singular right R-modules are flat. It is proved that a ring R is strongly regular if and only if R is a semiprime right quasi-duo ring whose simple singular right R-modules are flat. Moreover, it is shown that if R is a P. I.-ring whose simple singular right R-modules are flat, then R is a strongly $\pi$-regular ring. Finally, it is shown that if R is a right continuous ring whose simple singular right R-modules are flat, then R is a von Neumann regular ring.

Key words: semiprime , flat ,regular ring .

**حول الحلقات التي فيها كل مقاس بسيط منفرد مسطح , I**

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**الملخص**

في هذا البحث نختبر الحلقات المنتظمة عندما يكون لدينا كل موديول بسيط منفرد أيمن فيها مسطحا. من النتائج ان R منتظمة بمفهوم فون نيومان اذا وفقط اذا كانت R شبه أولي وشبه . ديو اليمنى وان كل موديول بسيط منفرد أيمن يكون فيها مسطحا. وكذلك بينا اذا كانت R من نوع P.I. وكل موديول بسيط منفرد أيمن يكون فيها مسطحا فان R حلقة منتظمة بقوة من النمط τ . وأخيرا بينا أنه اذا كانت R حلقة مستمرة يمنى وأن موديول بسيط منفرد أيمن يكون فيها مسطحا فان R حلقة منتظمة بمفهوم فون نيومان.

الكلمات المفتاحية: شبه اولي , مسطح , حلقة منتظمة.

### 1. Introduction

Throughout this paper, R denotes an associative ring with identity and all modules are unitary right R-modules. For any nonempty subset X of a ring R, r(X) and $\ell$(X) denote the right and left annihilators of X, respectively. Following [14], for any ideal I of R, R/I is flat if and only if for each $a \in I$, there exists $b \in I$ such that a=ba. Also, it was proved that a ring R is strongly regular if and only if it is a right quasi-duo ring whose simple right R-module are flat [14]. Indeed, Mahmood R. D. and Ibrahim Z. M., proved that a ring R is strongly regular if and only if R is ZC ring whose simple singular right R-modules are flat [8]. Moreover we investigate regularity of rings whose simple singular right R-module are flat. We recall that:

(1) A ring R is called reduced if it contains no nonzero nilpotent elements.

(2) R is said to be von Neumann regular (or just regular) if for every $a\in R$, $a\in aRa$ and R is called strongly regular [7] if $a\in a^2R$.

Clearly every strongly regular ring is reduced regular. Y(R), J(R), Rad(R), N(R) and Cent(R) will be stand for the right singular ideal, Jacobson radical, the prime radical, the set of all nilpotent elements and the center of R, respectively.

(3) A ring R is called 2-primal ring [2, 3] if Rad(R) = N(R), or equivalently, if R / Rad(R) is a reduced ring.

(4) A ring R is called a P. I.-ring [1] if R satisfies a polynomial identity with coefficients in the centroid and at least one coefficient is invertible.

(5) A ring R is called semi-primitive [14] if it's Jacobson radical is zero.

## 2. Regularity of Rings Whose Simple Singular R-Modules Are Flat

In this section we investigate von Neumann regularity of rings whose simple singular right R-modules are flat.

We start this section with the following definition.

**Definition 2.1:**

A ring R is called right (left) quasi-duo [14] if every maximal right(left) ideal of R is a two-sided ideal.

Rege in [14], proved that a right quasi-duo ring whose simple right R-modules are flat is strongly regular.

The following results are given in [14].

**Lemma 2.2:**

Let R be a reduced ring. Then, R is a left weakly regular ring if and only if R is a right weakly regular ring.

**Lemma 2.3:**

A semi-primitive quasi-duo ring is reduced.

**Lemma 2.4:**

A ring R has zero prime radical if and only if it contains no nonzero nilpotent ideal.

**Proof:**

See [9, Theorem 4.25]. ♦

Now, we give the following definition see in [14] and [8].

**Definition 2.5:**

A ring R is said to be a right SF-ring (SSF-ring) if and only if every simple(simple singular) right R-modules are flat.

**Theorem 2.6:**

Let R be a semiprime, 2-primal and SSF-ring. Then, R is a weakly regular ring.

**Proof:**

Let $0 \neq a \in R$ such that $a^2 = 0$. Thus $a \in Rad(R)$. Since R is semiprime, 2-primal ring, then it has no nonzero nilpotent ideal and by Lemma 2.4, Rad(R) = 0, so a = 0 and hence R is reduced.

Now, we will show that RaR+r(a)=R, for any $a \in R$. If not, then there exists a maximal right ideal M of R such that $RbR+r(b) \subseteq M$, for some $b \in R$. So, M is an essential right ideal of R. Since R/M is flat, there exists c $\in R$ such that b=cb. This implies that $(1-c) \in \ell(b) = r(b) \subseteq M$, yielding $1 \in M$, which is a contradiction. Therefore, R is a right weakly regular ring. Since R is reduced. Then, by Lemma 2.2, R is a left weakly regular. ♦

**Corollary 2.7:**

A semi-primitive, quasi duo and SSF-ring is a weakly regular ring.

The following theorem extends Theorem 4.10 of [14].

**Theorem 2.8:**

The following statements are equivalent:

(1) R is a strongly regular ring;
(2) R is a right quasi-duo and SF-ring;
(3) R is a semiprime right quasi-duo and SSF-ring.

**Proof:**

Obviously (1) implies (2), follows from [14, Theorem 4.10]. Also using [14, Theorem 4.10] and [3, Proposition 1.5.5] to prove (2) implies (3).

(3) implies (1). First we will show that R is reduced. Assume that $0 \neq a \in R$ such that $a^2 = 0$. Then, there exists a maximal right ideal M of R such that $RaR+r(a) \subseteq M$. Observe that M must be an essential right ideal of R. For if M is not essential, then we can write $M = r(e)$, where $0 \neq e = e^2 \in R$. Since $eRaR = 0$ and R is semiprime, we have $aReR = 0$. Thus, $e \in r(a) \subseteq M = r(e)$, whence $e = 0$. It is a contradiction. Therefore, R/M is flat. Then, there exists $b \in M$ such that $a = ba$ and hence $(1-b)a = 0$. Thus, $(1-b) \in M$, whence $1 \in M$ which is a contradiction. Therefore, R is reduced. Now, combining this with Theorem 2.5 in [14], we obtain that R is a strongly regular ring. ♦

Recall that a ring R is said to be right strongly prime [5] if every nonzero ideal of R contains a finite subset F such that $r(F) = 0$.

**Lemma 2.9:**

Every strongly prime ring R is nonsingular.

**Proof:**

See [11, Proposition 2.2]. ♦

**Theorem 2.10:**

Let R be a strongly prime and SSF-ring. Then, Cent(R) is strongly regular ring.

**Proof:**

First we have to prove that Cent(R) is reduced. Let $0 \neq a \in Cent(R)$ and $a^2 = 0$ implies that $a \in r(a)$. If r(a) is an essential ideal, then $a \in Y(R) = 0$ implies that $a = 0$. We are done. If r(a) is not essential ideal, there exists a right ideal I of R such that $r(a) \cap I = 0$ and $I \neq 0$. Then, $Ia \subseteq I \cap r(a)$, but $I \cap r(a) = 0$ implies $Ia = 0$ and hence we obtain that $I \subseteq \ell(a) = r(a)$, so $I = 0$, a contradiction. Therefore, $a = 0$, so Cent(R) is a reduced ring.

Now, we shall show that $aR + r(a) = R$, for any $a \in Cent(R)$. If not, there exists a maximal right ideal M of R such that $aR + r(a) \subseteq M$, observe that M is an essential right ideal of R. If not, then M is a direct summand of R. So, we can write $M = r(e)$, for some $0 \neq e = e^2 \in R$. Since $a \in M$ and $a \in Cent(R)$, then $ae = ea = 0$. Thus, $e \in r(a) \subseteq M = r(e)$, whence $e = 0$. It is a contradiction. Therefore M must be an essential right ideal of R. Since R/M is flat, there exists $b \in M$ such that $a = ba$. Hence, $1 \in M$ which is a contradiction. Therefore, $aR + r(a) = R$, for any $a \in Cent(R)$ and we have $a = ara$, for some $r \in R$. If we set $d = a^2r^3$, where $d \in Cent(R)$, then $ada = a(a^2r^3)a = ararara = ara = a$. Now, consider

$(a-a^2d)^2 = a^2-a^3d-a^2da+a^2da^2d = a^2 - a^3d - a^2 + a^3d = 0$. But Cent(R) is reduced, then $a- a^2d = 0$, gives $a = a^2d$. Therefore, Cent(R) is strongly regular ring. ♦

**Lemma 2.11:**

Let R be a P.I.-ring. If every prime factor ring of R is a right weakly regular ring, then R is a strongly π-regular ring.

**Proof:**

See [1, Theorem 1] and [4, Theorem 2.3]. ♦

**Theorem 2.12:**

Let R be a P.I.and SSF-ring. Then, R is a strongly π-regular ring.

**Proof:**

By Lemma 2.11, it is enough to show that every prime factor ring of R is right weakly regular. Assume that $\bar{R} = R/P$ is not right weakly regular, for some prime ideal P of R. Then, there exists $\bar{a} \in \bar{R}$ such that $\bar{R}\,\bar{a}\,\bar{R} \neq \bar{R}$. Let $\bar{M} = M/P$ be a maximal right ideal of $\bar{R}$ containing $\bar{R}\,\bar{a}\,\bar{R}$. Observe that M must be a maximal essential right ideal of R. For, if M is not essential, then we can write M = r(e), where $0 = e = e^2 \in R$. Thus, $eRaR = 0 \in P$ and $a \notin P$, whence $e = 0$, which is a contradiction. Since R/M is flat, then for every $a \in M$, there exists $y \in M$ such that $a = ya$. Thus, $(1-y) \in M$, whence $1 \in M$ which is a contradiction. Hence every prime factor ring of R is right weakly regular. Whence R is str-
ngly π-regular. ♦

Finally, we investigate the von Neumann regularity of right continuous rings whose simple singular right R-modules are flat. Recall that a ring R is right continuous if it satisfies the following conditions:

(1) For any right ideal I of R, there is an idempotent e such that eR is an essential extention of I;

(2) If fR, $f^2 = f$ is isomorphic to a right ideal J, then J also is generated by an idempotent [15].

Utumi introduced the following result in [15].

**Lemma 2.13:**

If R is a right continuous ring, then Y(R) = J(R) and R/J(R) is a von Neumann regular ring.

**Theorem 2.14:**

Let R be a right continuous and SSF-ring. Then, R is a von Neumann regular ring.

**Proof:**

It suffices to show that Y(R) = 0 by Lemma 2.13. Suppose that Y(R) $\neq$ 0, there exists a nonzero element a $\in$Y(R) such that $a^2 = 0$ [10]. Then, there exists a maximal right ideal M of R containing Y(R)+ r(a), whence M is an essential. Thus, by hypothesis R/M is flat. Therefore, there exists b$\in$M such that a = ba and so (1-b) $\in$ M; whence 1 $\in$ M which is a contradiction. Therefore, Y(R)+ r(a) = R, so there exists x $\in$ Y(R) and y $\in$ r(a) such that x + y = 1. We multiply a on the left hand side, we obtain ax+ay = a and so a(1-x) = 0. By Lemma 2.13, x $\in$ Y(R) = J(R). Thus, 1-x is invertibe and hence a = 0, which is also a contradiction. Whence R is a von Neumann regular. ♦

**Theorem 2.15:**

Let R be a P.I., right continuous and SSF-ring. Then, the Jacobson radical of R is a nilideal.

**Proof:**

By Lemma 2.13, Y(R) = J(R). Suppose that J(R) is non nilideal. Then, there exists y $\in$ Y(R) = J(R) such that $y^m \neq 0$, for all positive integer m. Since R is a P.I. and SSF-ring, then by Theorem 2.12, R is strongly $\pi$-regular and hence $y^n = d\, y^{n+1}$, for some d $\in$ R. Now, r(dy) $\cap$ $y^nR = 0$ and since dy $\in$ Y(R), then $y^n = 0$, a contradiction. This proves that J(R) is a nilideal of R. ♦

Recall that R is called right Goldie ring if and only if R has ascending chain condition on both complement right ideal and annihilator right ideals.

**Theorem 2.16:**

Let R be a P.I., semiprime, right Goldie and SSF-ring. Then, R is a semisimple Artinian ring.

**Proof:**

By Theorem 2.12, R is strongly $\pi$-regular. Let I be an essential right ideal of R. Therefore, by Theorem 1.10 in [3], I contains a regular element c $\in$ R. Since R is strongly $\pi$-regular, there exists a positive integer n such that $c^n = c^{n+1}d$, for some d $\in$ R. So, $c^n$ (1-cd) = 0 and hence 1 = cd $\in$ I. Therefore, I = R and so R is a semisimple Artinian ring. ♦

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