On $S\pi$ – Weakly Regular Rings, II

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ABSTRACT

The main purpose of this paper is to study right(left) $S\pi-$Weakly regular rings. also we give some properties of $S\pi-$Weakly regular rings, and the connection between such rings and CS-rings, MGP-rings and SSGP-rings.

Keywords: $S\pi-$Weakly regular rings, CS-rings, MGP-rings, SSGP-rings.

الحلقات المنتظمة الضعيفة من النمط – $S\pi -$

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الملخص

الهدف الرئيسي من هذا البحث هو دراسة الحلقات المنتظمة الضعيفة من النمط– $S\pi$.
بالإضافة إلى بعض الخواص الأساسية لهذه الحلقات وعلاقاتها مع الحلقات الأخرى مثل الحلقات من النمط –CS ، الحلقات من النمط –MGP والحلقات من النمط –SSGP .

الكلمات المفتاحية: الحلقات المنتظمة الضعيفة من النمط– $S\pi$، الحلقات من النمط –CS ، الحلقات من النمط –MGP، الحلقات من النمط –SSGP .

1- Introduction

Throughout this paper. R is an associative ring with identity. A ring R is said to be right(left) **S-weakly regular ring** if for each $a \in R$, $a \in aRa^2R\,(a \in Ra^2Ra)$.

This concept was introduced by W.B. Vasantha kandasamy [7]. As a generalization of this concept the authors in [3] defined $S\pi-$**Weakly regular ring** that is a ring such that for each $a \in R$, there exists a positive integer n, $a^n \in a^n R a^{2n} R (a^n \in R a^{2n} R a^n)$. In the present work we develop further properties of $S\pi-$Weakly regular rings, and we give the connection of $S\pi-$Weakly regular rings with other rings.

Recall that:-

1- An ideal I of a ring R is called **right(left) GP- ideal** if for every $a \in R$, there exists $b \in R$ and a positive integer n such that $a^n = a^n b (a^n = b a^n)$. [6]
2- A ring R is said to be a **right MGP-ring** if and only if every maximal right ideal is left GP-ideal.[6]
3- R is called **reduced** if it has no non nilpotent elements.
4- According to cohn [1], a ring R is called **reversible** if $ab = 0$ implies $ba = 0$ for $a, b \in R$. It is easy to see that R is reversible if and only if right (left) annihilator of a in R is two sided ideal.
5- A ring R is said to be **right SSGP-ring** if every simple singular right R-module is GP- injective [4].
6- $J(R)$ denote the Jacobson radical.

2- $S\pi-$Weakly regular rings

In this section we give some of basic properties, as well as a connection between CS-rings, MGP-rings, SSGP-rings and $S\pi-$Weakly regular rings.

begins with the following definition .

Definition 2.1:[3]

R is called **right (left) $S\pi-$Weakly regular ring** if for each $a \in R$, there exists a positive integer $n = n(a)$ depending on a such that $a^n \in a^n R a^{2n} R (a^n \in R a^{2n} R a^n)$. R is called $S\pi-$**Weakly regular ring** if it is both right and left $S\pi-$Weakly regular ring.

Theorem2.2:

Let R be $S\pi-$Weakly regular ring and $a^n R = R a^n$ with $r(a^n) \subseteq r(a)$ for every $a \in R$ and a positive integer n, then $J(R) = (0)$.

Proof:

Let $0 \neq a \in J(R)$, if $aR + r(a) \neq R$, then there exists a maximal right ideal M containing $aR + r(a)$, since R is $S\pi-$Weakly regular ring, then there exists $b,c \in R$ and a positive integer n such that $a^n = a^n b a^{2n} c$, so $a^n(1-ba^{2n}c)=0$ implies that $(1-ba^{2n}c) \in r(a^n) \subseteq r(a) \subseteq M$ so $1 \in M$ a contradiction. Therefore $aR + r(a) = R$, in particular $ab + d = 1$ for some $b \in R, d \in r(a)$, so $a^2 b = a$ implies $a(1-ab)=0$ since $a \in J(R)$, so there exists an invertible element v in R such that $(1-ab)v = 1$ multiply in the left by a, $(a - a^2 b)v = a$ implies $a = 0$. Therefore $J(R) = (0)$.

Theorem 2.3:

Let R be a ring with out identity and with out divisors of zero. Then R is $S\pi -$Weakly regular ring if and only if for every $a \in R$ there exists $b,c \in R$ and a positive integer n with $a^n = a^n b a^{2n} c$.

Proof:

Given a ring R with out identity and with out divisors of zero. Let R be $S\pi -$Weakly regular ring. Then for every $a \in R$ we have $a^n \in a^n R a^{2n} R$, thus $a^n = a^n b a^{2n} c$ for some $b,c \in R$.

Conversely, if $a^n = a^n b a^{2n} c$ for every $a \in R$, and a positive integer n we have obviously R to the $S\pi -$ Weakly regular ring and given R has no identity and zero divisors.

The following result is a connection between MGP- ring and $S\pi -$Weakly regular ring by adding reversible condition.

Theorem 2.4:

Let R be a right MGP-ring and reversible ring. Then R is a right $S\pi -$Weakly regular ring.

Proof:

Let R be a right MGP- ring, to prove R is a right $S\pi -$Weakly regular ring, let $Ra^{2n}R + r(a^n) = R$, if not then there exists a maximal right ideal M containing $Ra^{2n}R + r(a^n)$ such that $Ra^{2n}R + r(a^n) \subseteq M$, since R is a right MGP- ring, then every maximal right ideal of R is left GP-ideal. Then for all $a \in M$, there exists $b \in M$ and a positive integer n such that $a^n = ba^n$ then $a^n - ba^n = 0$, implies $(1-b)a^n = 0$ and $(1-b) \in l(a^n) = r(a^n) \subseteq M$ (R is reversible ring), hence $1 \in M$, a contradiction, hence $Ra^{2n}R + r(a^n) = R$. Then $ba^{2n}c + d = 1$, for some

$b,c \in R$ and $d \in r(a^n)$ implies that $a^n ba^{2n} c + a^n d = a^n$. Therefore $a^n \in a^n Ra^{2n} R$, so R is a right $S\pi -$Weakly regular ring.

Example:

Let Z_{12} be the ring of integers module 12, then the maximal ideals, $I = \{o,3,6,9\}$, $J = \{o,2,4,6,8,10\}$ are GP-ideal, hence Z_{12} is a right MGP-ring and a right $S\pi -$Weakly regular ring.

Definition 2.5:[2]

A ring R is said to be **right (left) CS- ring** if every non zero right (left) ideal is essential in a direct summand.

The next result gives a sufficient condition for CS- ring to be $S\pi -$Weakly regular ring.

Theorem 2.6:

Let R be a reversible right CS- ring and $r(a^{2n}) \subseteq r(a^n)$ for every $a \in R$ and a positive integer n. Then R is a $S\pi -$Weakly regular ring.

Proof:

Let $0 \neq a \in R$ such that $Ra^{2n} R + r(a^n) = R$, if not then there exists a maximal right ideal M containing $Ra^{2n} R + r(a^n)$ since M is a direct summand, there exists K right ideal such that $M \oplus K = R$, this meaning $Ra^{2n} R + r(a^n) \cap K = 0$, which implies that $Ra^{2n} R K \subseteq MK \subseteq M \cap K = 0$, then $K \in r(a^{2n}) \subseteq r(a^n)$, hence $K \subseteq r(a^n) \subseteq M$ but $M \cap K = 0$, contradiction, if $Ka \neq 0$, $Ka \subseteq M \cap K = 0$ implies that $Ka = 0$, contradiction, hence $Ra^{2n} R + r(a^n) = R$, in particular $ra^{2n} s + t = 1$, where $r,s \in R$ and $t \in r(a^n)$, so $ra^{2n} sa^n + ta^n = a^n$ henc1e $ta^n = 0$ (since $t \in r(a^n) = l(a^n)$ R is reversible ring), implies $ra^{2n} sa^n = a^n$, then $a^n \in Ra^{2n} Ra^n$ as well as $a^n ra^{2n} s + a^n t = a^n$, so $a^n \in a^n Ra^{2n} R$. Therefore R is a $S\pi -$Weakly regular ring.

The following result is due to S.B. Nam [4].

Lemma 2.7:[4]

Let R be a SSGP- ring and $l(a) \subseteq r(a)$ for every $a \in R$. Then R is a reduced ring.

Proposition 2.8:

Let R be a SSGP- ring and $l(a) \subseteq r(a)$ for every $a \in R$. Then R is $S\pi -$Weakly regular ring.

Proof:

Assume that R is SSGP- ring and $l(a) \subseteq r(a)$ for every $a \in R$, then by Lemma 2.8, R is reduced ring. We will show that $Ra^{2n}R + r(a^n) = R$, for any $a \in R$ and a positive integer n. Suppose that $Ra^{2n}R + r(a^n) \neq R$. Then there exists a maximal right ideal M of R containing $Ra^{2n}R + r(a^n)$. Thus R/M is GP- injective, so any R- homomorphism of $a^{2n}R \to R/M$ extends to one of R into R/M. Let $f : a^{2n}R \to R/M$ be defined by $f(a^{2n}t) = t + M$, for all $t \in R$. Since R is reduced ring, then f is a well-defined R- homomorphism. Now, R/M is GP-injective. So there exists $c \in R$ such that $1 + M = f(a^{2n}) = ca^{2n} + M$. Hence $1 - ca^{2n} \in M$ and so $1 \in M$, which is a contradiction. Therefore $Ra^{2n}R + r(a^n) = R$. In particular, $1 = ba^{2n}d + x$ for some $b, d \in R$ and $x \in r(a^n)$. Therefore $a^n = a^n ba^{2n}d$ .So $a^n R = a^n Ra^{2n}R$ and hence R is $S\pi -$Weakly regular ring.

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