Solution of Eighth Order Boundary Value Problems Using Differential Transformation Technique

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ABSTRACT

In this paper we have applied the Differential Transform Method (DTM) for solving eighth-order boundary value problems. The analytical and numerical results of the equations have been obtained in terms of convergent series with the easily computable components. Three examples are considered for the numerical illustrate and implementation of this method. Numerical Comparisons with respect to the analytical solutions have been considered . It is observed that the method is an alternative and efficient for finding the approximate solutions of the boundary values problems.

Key Words: Differential Transform Method**,** boundary value problems, analytical solution

حل مسائل القيم الحدودية من الرتبة الثامنة باستخدام طريقة التحويل التفاضلي

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الملخص

تم في هذا البحث استخدام طريقة التحويل التفاضلي لحل مسائل القيم الحدودية من الرتبة الثامنة، وقد تم الحصول على النتائج التحليلية والعددية للمعادلات من خلال متسلسلات متقاربة ذات معاملات محددة بطرق حسابية سهلة. تضمن البحث ثلاث أمثلة لتطبيق الطريقة وتوضيح الطريقة مع الحلول التحليلية ومن خلال هذه نتائجها العددية. المقارنات العددية قد أجريت لهذه المقارنات تبين دقة وكفاءة هذه الطريقة في إيجاد الحلول التقريبية لمسائل القيم الحدودية هذه.

الكلمات المفتاحية: طريقة التحويل التفاضلي, مسائل القيم الحدودية, الحل التحليلي.

1. Introduction

In this paper, we considered the general eighth-order boundary value problems of the type

$$y^{(8)}(x) + f(x)y(x) = g(x), \qquad x \in [a,b] \qquad \ldots(1)$$

with boundary conditions

$$y(a)=\alpha_1\,,\ \ y^{(1)}(a)=\beta_1\,,\ \ y^{(2)}(a)=\xi_1\,,\ \ y^{(3)}(a)=\sigma_1$$
$$y(b)=\alpha_2\,,\ \ y^{(1)}(b)=\beta_2\,,\ \ y^{(2)}(b)=\xi_2\,,\ \ y^{(3)}(b)=\sigma_2 \qquad \ldots(2)$$

Where $\alpha_i\,,\beta_i\,,\xi_i$ and σ_i, $\mathrm{i}=1,2$ are the finite real constants while the functions $f(x)$ and $g(x)$ are continuous on $[a,b]$.

A class of characteristic-value problems of higher order (as higher as twenty four) is known to arise in hydrodynamic and hydro magnetic stability problems. When an infinite horizontal layer of fluid is heated from below and is subject to the action of rotation, instability sets in. When this instability is as ordinary convection the ordinary differential equation is sixth order, when the instability sets in as over stability, it is modeled by an eighth order ordinary differential equation with appropriate boundary value conditions (see Chandrasekhar [6]). The boundary value problems of higher order have been investigated because of both of their mathematical importance and the potential for applications in hydrodynamic and hydro magnetic stability. Agarwal [1] presented the theorems which listed the conditions for the existence and uniqueness of solutions of eighth-order BVPs problems. Scott and Watts [21] developed a numerical method for the solution of linear BVPs using a combination of superposition and orthonormalization. Scott and Watts [20] described several computer codes that were developed using the superposition and orthonomalization technique and invariant imbedding. Twizell et al. [5] developed numerical methods for eighth, tenth and twelfth order Eigen value problems arising in thermal instability Problems. Boutayeb and Twizell [4] developed finite difference methods for the solution of eighth-order BVPs. Siddiqi and Twizell [19] presented the solution of eighth-order BVPs using octic spline. Siddiqi and Akram [17,18] presented the solutions of eighth-order linear special case BVPs using nonic spline and nonpolynomial nonic spline respectively. Inc and Evans [9] presented the solutions of eighth-order BVPs using Adomian decomposition method.

Liu and Wu [13] presented differential quadrature solutions of eighth-order differential equations. Siddiqi, Akram and Zaheer [16] presented the solutions of eighth-order BVPs using Variational Iteration Technique.

In this paper, we employed differential transform method to solve Eq.(1) with boundary conditions (2).The concept of differential transform was first introduced by Zhou [22], in a study about electric circuit analysis. It is a semi numerical-analytic-technique that formulizes Taylor series in a totally different manner. With this method, the given differential equation and related boundary conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. This method is useful to obtain exact and approximate solutions of linear and nonlinear differential equations .no need to linearization or discretization, large computational work and round-off errors are avoided. It has been used to solve effectively, easily and accurately a large class of linear and nonlinear problems with approximations. The method is well addressed in [2, 3, 7, 10, 11, 14, 22].

2. Differential Transformation Method (DTM).

In order to solve the boundary value problems (1) - (2) by (DTM), its basic definitions are stated briefly in this section as follows:

Definition 2.1 [8, 15] If $f(x)$ is analytic in domain D, Let $x = x_0$ represent any point within domain D, thus the differential transformation of $f(x)$ is given by

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \qquad \ldots(3)$$

Where $f(x)$ is the original function and $F(k)$ is the transformed function.

Definition 2.2 [8, 15] If $f(x)$ can be represented by Taylor's series, then it can be represented as

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0} .(x - x_0)^k = \sum_{k=0}^{\infty} F(k)(x - x_0)^k \qquad \ldots(4)$$

Eq. (4) is known as inverse transformation of $F(k)$.

In real application the function $f(x)$ is expressed by finite series and Eq. (4) can be written as

$$f(x) = \sum_{k=0}^{n} F(k)(x - x_0)^k \qquad \ldots(5)$$

Eq. (5) implies that $\sum_{k=n+1}^{\infty} F(k)(x - x_0)^k$ is negligibly small.

From Eq. (3) and (4), it is easily proven that the transformation function have basic mathematics operations shown in Table 1 [15].

Table 1. The fundamental operations of differential transformations method

Original function	**Transformed function**
$y(x)=u(x)\pm v(x)$	$Y(k)=U(k)\pm V(k)$
$y(x)=\alpha\, u(x)$	$Y(k)=\alpha\, U(k)$
$y(x)=\dfrac{d^m u(x)}{dx^m}$	$Y(k)=\dfrac{(k+m)!}{k!}U(k+m)$
$y(x)=u(x).v(x)$	$Y(k)=\sum_{l=0}^{k}U(l)V(k-l)$
$y(x)=u_1(x)u_2(x)....u_n(x)$	$Y(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}...\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}U_2(k_1)U_1(k_2-k_1)\times ....U_{n-1}(k_{n-1}-k_{n-2})U_n(k-k_{n-1})$
$y(x)=x^m$	$Y(k)=\delta(k-m)=\begin{cases}1, & if\ k=m\\ 0, & if\ k\neq m\end{cases}$
$y(x)=\exp(\lambda x)$	$Y(k)=\dfrac{\lambda^k}{k!}$
$y(x)=\sin(wx+\alpha)$	$Y(k)=\dfrac{w^k}{k!}\sin(\dfrac{k\pi}{2}+\alpha)$
$y(x)=\cos(wx+\alpha)$	$Y(k)=\dfrac{w^k}{k!}\cos(\dfrac{k\pi}{2}+\alpha)$

3. Applications and Numerical Results.

Example 1 Consider the following linear boundary value problem

$$y^{(8)}(x)=y(x)-48e^x-16xe^x \quad , \qquad x\in[0,1] \qquad \ldots(6)$$

with boundary conditions

$$\begin{aligned} &y(0)=0,\ \ y^{(1)}(0)=1,\ \ y^{(2)}(0)=0,\ \ y^{(3)}(0)=-3\\ &y(1)=0,\ \ y^{(1)}(1)=-e,\ \ y^{(2)}(1)=-4e,\ \ y^{(3)}(1)=-9e \end{aligned} \qquad \ldots(7)$$

The exact solution of the problem [16] is:

$$y(x)=x(1-x)e^x \qquad \ldots(8)$$

Solution:

Taking the differential transform of both sides of Eq. (6), we obtain

$$Y(k+8) = \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)} \times \left(Y(k) - \frac{48}{k!} - 16\sum_{l=0}^{k} \frac{\delta(l-1)}{(k-l)!} \right) \qquad \ldots(9)$$

Using Eqs. (4) and (7), the following transformed boundary conditions at $x_0 = 0$ can be obtained:

$$Y(0)=0,\ Y(1)=1,\ Y(2)=0,\ Y(3)=-\frac{1}{2},\ \sum_{k=0}^{n} Y($$

$$\sum_{k=0}^{n} k(k-1)Y(k) = -4e,\ \sum_{k=0}^{n} k(k-1)(k-2)Y(k) = -9e \qquad \ldots(10)$$

Utilizing the recurrence relation in Eq.(9) and the transformed boundary conditions in Eq.(10),the following series solution up to $O(x^{17})$ is obtained:

$$\begin{aligned} y(x) = {} & x - \frac{x^3}{2} + Ax^4 + Bx^5 + Cx^6 + Dx^7 - \frac{x^8}{840} - \frac{x^9}{5760} - \frac{x^{10}}{45360} \\ & - \frac{x^{11}}{403200} + \left(\frac{A}{19958400} - \frac{1}{4276800} \right) x^{12} + \left(\frac{B}{51891840} \right. \\ & \left. - \frac{1}{48648600} \right) x^{13} + \left(\frac{C}{121080960} - \frac{1}{605404800} \right) x^{14} \\ & + \left(\frac{D}{259459200} - \frac{1}{8172964800} \right) x^{15} + \frac{x^{16}}{9340531000} + O(x^{17}) \end{aligned} \qquad \ldots(11)$$

and, according to Eq.(3),

$$A = \frac{y^{(4)}(0)}{4!} = Y(4),\ B = \frac{y^{(5)}(0)}{5!} = Y(5),\ C = \frac{y^{(6)}(0)}{6!} = Y(6)$$

$$D = \frac{y^{(7)}(0)}{7!} = Y(7). \qquad \ldots(12)$$

By taking $n = 16$,the following system of equations can be obtained from Eq. (10):

$$\frac{19958401}{19958400}A + \frac{51891841}{51891840}B + \frac{121080961}{121080960}C + \frac{259459201}{259459200}D = -\frac{108670165831}{217945728000}$$

$$\frac{6652801}{1663200}A + \frac{19958401}{3991680}B + \frac{51891841}{8648640}C + \frac{121080961}{17297280}D = -\left(e - \frac{83582810233}{163459296000}\right)$$

$$\frac{1814401}{151200}A + \frac{6652801}{332640}B + \frac{19958401}{665280}C + \frac{51891841}{1235520}D = -\left(4e - \frac{11992689455}{3891888000}\right)$$

$$\frac{362881}{15120}A + \frac{1814401}{30240}B + \frac{6652801}{55440}C + \frac{19958401}{95040}D = -\left(9e - \frac{436660650720}{124540416000}\right) \quad ...(13)$$

From Eq.(13), A, B, C and D are evaluated numerically as

$$A = -0.33333333316155, \quad B = -0.12500000055789$$
$$C = -0.03333333272519, \quad D = -0.00694444466725 \quad \ldots(14)$$

Then Eq.(11) becomes

$$y(x) = x - 0.5x^3 - 0.33333333316155x^4 - 0.12500000055789x^5 - 0.03333333272519x^6 - 0.00694444466725x^7 - 0.00119047619048x^8 - 0.00017361111111x^9 - 0.00002204585538x^{10} - 2.48015873\times10^{-6}x^{11} - 2.5052108\times10^{-7}x^{12} - 2.296443\times10^{-8}x^{13} - 1.92709\times10^{-9}x^{14} - 1.4912\times10^{-10}x^{15} - 1.071\times10^{-11}x^{16} + O(x^{17}). \quad ...(15)$$

Tables 2 bellow exhibits the numerical results and the errors obtained by using the differential transform method (DTM) with comparison to the exact solution.

Table 2: The series solution and error estimation of DTM compared with exact solution.

x	Exact solution	Series solution	*Errors
0.0	0.00000000000000	0.00000000000000	0.00
0.1	0.09946538262681	0.09946538262682	-1.00E-14
0.2	0.19542444130563	0.19542444130576	-1.30E-13
0.3	0.28347034959096	0.28347034959139	-4.30E-13
0.4	0.35803792743390	0.35803792743472	-8.20E-13
0.5	0.41218031767503	0.41218031767610	-1.07E-12
0.6	0.43730851209372	0.43730851209474	-1.02E-12
0.7	0.42288806856880	0.42288806856948	-6.80E-13
0.8	0.35608654855879	0.35608654855906	-2.70E-13

***Error=Exact solution-Series solution**

From the above table, we can conclude that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of $y(x)$.

Example 2 For $x \in [0,1]$, the following nonlinear eighth-order boundary value problem are considered

$$y^{(8)}(x) = e^{-x} y^2(x) \qquad \text{...(16)}$$

with boundary conditions

$$\begin{aligned} &y(0)=1,\ \ y^{(1)}(0)=1,\ \ y^{(2)}(0)=1,\ \ y^{(3)}(0)=1 \\ &y(1)=e,\ \ y^{(1)}(1)=e,\ \ y^{(2)}(1)=e,\ \ y^{(3)}(1)=e \end{aligned} \qquad \text{…(17)}$$

The analytical solution for this problem [16] is:

$$y(x) = e^{x} \qquad \text{...(18)}$$

Solution:

Taking the differential transform of both sides of Eq. (16), we obtain the following recurrence relation:

$$Y(k+8) = \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)} \times \left(\sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} \frac{(-1)^{k-k_2}}{(k-k_2)!} Y(k_1) Y(k_2-k_1) \right) \qquad \text{...(19)}$$

The boundary conditions in Eq. (17) can be transformed at $x_0 = 0$ as follows:

$$Y(0)=1,\ Y(1)=1,\ Y(2)=\frac{1}{2},\ Y(3)=\frac{1}{6},\ \sum_{k=0}^{n}Y(k)=e,\ \sum_{k=0}^{n}kY(k)=e$$

$$\sum_{k=0}^{n}k(k-1)Y(k)=e,\ \sum_{k=0}^{n}k(k-1)(k-2)Y(k)=e. \qquad ...(20)$$

Utilizing the recurrence relation in Eq.(19) and the transformed boundary conditions in Eq.(20), the following series solution up to $O(x^{14})$ is obtained

$$y(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}+Ax^4+Bx^5+Cx^6+Dx^7+\frac{x^8}{40320}+\frac{x^9}{362880}$$
$$+\frac{x^{10}}{3628800}+\frac{x^{11}}{39916800}+\left(\frac{A}{9979200}-\frac{1}{479001600}\right)x^{12}+\left(\frac{B}{25945920}\right.$$
$$\left.-\frac{1}{6227020800}\right)x^{13}+O(x^{14}). \qquad ...(21)$$

Taking $n=13$ and using Eqs. (19) and (20), we can obtain the following system of equations:

$$\frac{9979201}{9979200}A+\frac{25945921}{25945920}B+C+D=e-\frac{16605562258}{6227020800}.$$
$$\frac{3326401}{831600}A+\frac{9979201}{1995840}B+6C+7D=e-\frac{1197612359}{479001600}.$$
$$\frac{907201}{75600}A+\frac{3326401}{166320}B+30C+42D=e-\frac{79898048}{39916800}. \qquad \ldots(22)$$
$$\frac{181441}{7560}A+\frac{907201}{15120}B+120C+210D=e-\frac{3664879}{3628800}.$$

We get from the above system:

$$A=0.04166666560318,\quad B=0.00833333687487$$
$$C=0.00138888490941,\quad D=0.00019841421213 \qquad \ldots(23)$$

Then the series solution becomes

$$y(x) = 1 + x + 0.5x^2 + 0.16666666666667x^3 + 0.0416666660318x^4$$
$$+ 0.00833333687487x^5 + 0.00138888490941x^6$$
$$+ 0.00019841421213x^7 + 2.480158730\times10^{-5}x^8 + 2.75573192\times10^{-6}x^9$$
$$+ 2.7557319\times10^{-7}x^{10} + 2.505211\times10^{-8}x^{11} + 2.08768\times10^{-9}x^{12}$$
$$+ 1.6059\times10^{-10}x^{13} + O(x^{14}). \quad ...(24)$$

Table 3 bellow exhibits the numerical results and the errors obtained by using the differential transform method (DTM) with comparison to the exact solution.

Table 3: The series solution and error estimation of DTM compared with exact solution

x	Exact solution	Series solution	*Errors
0.0	1.00000000000000	1.00000000000000	0.00
0.1	1.10517091807565	1.10517091807557	8.00E-14
0.2	1.22140275816017	1.22140275815937	8.00E-13
0.3	1.34985880757600	1.34985880757342	2.58E-12
0.4	1.49182469764127	1.49182469763649	4.78E-12
0.5	1.64872127070013	1.64872127069398	6.15E-12
0.6	1.82211880039051	1.82211880038477	5.74E-12
0.7	2.01375270747048	2.01375270746676	3.72E-12
0.8	2.22554092849247	2.22554092849108	1.39E-12
0.9	2.45960311115695	2.45960311115680	.150E-13
1.0	2.71828182845905	2.71828182845905	0.00

***Error=Exact solution-Series solution**

For this table it is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of y(x).

Example 3 For $x \in [-1,1]$, let us consider the following boundary value problem

$$y^{(8)}(x) - y(x) = -8(2x\cos(x) + 7\sin(x)) \quad ...(25)$$

with the boundary conditions

$$y(-1) = y(1) = 0.$$
$$y^{(1)}(-1) = y^{(1)}(1) = 2\sin(1).$$
$$y^{(2)}(-1) = -y^{(2)}(1) = -4\cos(1) - 2\sin(1). \quad ...(26)$$
$$y^{(3)}(-1) = y^{(3)}(1) = 6\cos(1) - 6\sin(1).$$

The analytical solution of the above problem [12] is:

$$y(x) = (x^2 - 1)\sin(x). \qquad ...(27)$$

Solution:

By applying the fundamental mathematical operations performed by differential transform, the differential transform of Eq.(25) can be obtained as

$$Y(k+8) = \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)} \times \left(-16\sum_{k_1=0}^{k} \delta(k_1 - 1)C(k - k_1) - 56S(k) + Y(k) \right) \qquad ...(28)$$

Where $S(k)$ and $C(k)$ correspond to the differential transformation of $\sin(x)$ and $\cos(x)$ at $x_0 = 0$, respectively, which can be easily obtained from the definition of differential transform in Eq. (3) as follows:

$$S(k) = \begin{cases} \frac{(-1)^{\frac{(k-1)}{2}}}{k!}, & if\ k = odd \\ 0, & if\ k = even \end{cases} \qquad C(k) = \begin{cases} \frac{(-1)^{\frac{k}{2}}}{k!}, & if\ k = even \\ 0, & if\ k = odd \end{cases} \qquad \ldots(29)$$

The boundary conditions in Eq. (26) can be transformed at $x_0 = 0$ as

$$\sum_{k=0}^{n} Y(k)(-1)^k = 0,\ \sum_{k=0}^{n} Y(k) = 0,\ \sum_{k=0}^{n} kY(k)(-1)^{k-1} = 2\sin(1),\ \sum_{k=0}^{n} kY(k) = 2\sin(1),$$

$$\sum_{k=0}^{n} k(k-1)Y(k)(-1)^{k-2} = -4\cos(1) - 2\sin(1),\ \sum_{k=0}^{n} k(k-1)Y(k) = 4\cos(1) + 2\sin(1),$$

$$\sum_{k=0}^{n} k(k-1)(k-2)Y(k)(-1)^{k-3} = 6\cos(1) - 6\sin(1),$$

$$\sum_{k=0}^{n} k(k-1)(k-2)Y(k)(-1)^{k-3} = 6\cos(1) - 6\sin(1),$$

$$\sum_{k=0}^{n} k(k-1)(k-2)Y(k) = 6\cos(1) - 6\sin(1). \qquad ...(30)$$

where, n is a sufficiently large integer. By using the inverse transformation rule in Eq. (4), for $n = 9$ we get

$$y(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + A_5 x^5 + A_6 x^6 + A_7 x^7 + \frac{A_0}{40320} x^8 + \left(\frac{-1}{5040} + \frac{A_1}{362880} \right) x^9 + O(x^{10}). \quad ...(31)$$

where ,

$A_0 = y(0) = Y(0),\quad A_1 = y^{(1)}(0) = Y(1),\quad A_2 = y^{(2)}(0)/2! = Y(2),$

$A_3 = y^{(3)}(0)/3! = Y(3),\quad A_4 = y^{(4)}(0)/4! = Y(4),\quad A_5 = y^{(5)}(0)/5! = Y(5),$

$A_6 = y^{(6)}(0)/6! = Y(6),\quad A_7 = y^{(7)}(0)/7! = Y(7).$

Also, by taking $n = 9$, the following system of equations can be obtained from Eq.(30):

$$\begin{aligned}
&\frac{40321}{40320} A_0 - \frac{362881}{362880} A_1 + A_2 - A_3 + A_4 - A_5 + A_6 - A_7 = -\frac{1}{5040}.\\
&\frac{40321}{40320} A_0 + \frac{362881}{362880} A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 = \frac{1}{5040}.\\
&\frac{-1}{5040} A_0 + \frac{40321}{40320} A_1 - 2A_2 + 3A_3 - 4A_4 + 5A_5 - 6A_6 + 7A_7 = 2\sin(1) + \frac{1}{560}.\\
&\frac{1}{5040} A_0 + \frac{40321}{40320} A_1 + 2A_2 + 3A_3 + 4A_4 + 5A_5 + 6A_6 + 7A_7 = 2\sin(1) + \frac{1}{560}.\\
&\frac{A_0}{720} - \frac{A_1}{5040} + 2A_2 - 6A_3 + 12A_4 - 20A_5 + 30A_6 - 42A_7 = -4\cos(1) - 2\sin(1) - \frac{1}{70}.\\
&\frac{A_0}{720} + \frac{A_1}{5040} + 2A_2 + 6A_3 + 12A_4 + 20A_5 + 30A_6 + 42A_7 = 4\cos(1) + 2\sin(1) + \frac{1}{70}.\\
&\frac{-A_0}{120} + \frac{A_1}{720} + 6A_3 - 24A_4 + 60A_5 - 120A_6 + 210A_7 = 6\cos(1) - 6\sin(1) + \frac{1}{10}.\\
&\frac{A_0}{120} + \frac{A_1}{720} + 6A_3 + 24A_4 + 60A_5 + 120A_6 + 210A_7 = 6\cos(1) - 6\sin(1) + \frac{1}{10}.
\end{aligned} \quad \ldots(32)$$

We get from the equation system (32):

$$\begin{aligned}
&A_0 = 0,\ A_1 = -1.00001087423950,\ A_2 = 0,\ A_3 = 1.16670748209313,\\
&A_4 = 0,\ A_5 = -0.17505449473596,\ A_6 = 0,\ A_7 = 0.00855905534263.
\end{aligned} \quad ...(33)$$

Then, series solution becomes

$$y(x) = -1.00001087423950x + 1.16670748209313x^3 - 0.17505449473596x^5 + 0.00855905534263x^7 - 0.00020116846030x^9 + O(x^{11}). \quad ...(34)$$

By continuing the same procedure for $n = 15$, we get the following series solution:

$$y(x) = -0.99999999995027x + 1.16666666655206x^3 - 0.17499999987933x^5$$
$$+ 0.00853174598571x^7 - 2.0116843033\times10^{-4}x^9 + 2.78078403\times10^{-6}x^{11}$$
$$- 2.521270\times10^{-8}x^{13} + 1.613610^{-10}x^{15} + O(x^{17}). \qquad ...(35)$$

Table 4 and 5 bellow exhibit the numerical results and the errors obtained by using the differential transform method (DTM) with comparison to the exact solution for $n=9$ and $n=15$.

Table 4: The Comparison of numerical results of DTM w.r.t. the exact solution for n=9 and n=15 respectively

x	Exact solution	DTM (n=15)	DTM (n=9)
-1.0	0.00000000000000	-0.00000000001052	0.00000000000000
-0.8	0.25824819272383	0.25824819271282	0.25824836176569
-0.6	0.36137118297282	0.36137118295964	0.36137237446520
-0.4	0.32711140753927	0.32711140752555	0.32711365844677
-0.2	0.19072255756326	0.19072255755419	0.19072442297656
-0.0	0.00000000000000	0.00000000000000	0.00000000000000
0.2	-0.19072255756326	-0.19072255755419	-0.19072442297656
0.4	-0.32711140753927	-0.32711140752555	-0.32711365844677
0.6	-0.36137118297282	-0.36137118295964	-0.36137237446520
0.8	-0.25824819272383	-0.25824819271282	-0.25824836176569
1.0	0.00000000000000	0.00000000001052	0.00000000000000

Table 5: The error estimates of DTM when compared with the exact solution for n=9 and n=15 respectively

x	Exact solution	*Errors DTM (n=15)	*Errors DTM (n=9)
-1.0	0.00000000000000	1.052E-11	0.0000000
-0.8	0.25824819272383	1.100E-11	-1.6904186E-7
-0.6	0.36137118297282	1.318E-11	-1.19149238E-6
-0.4	0.32711140753927	1.372E-11	-2.2509075E-6
-0.2	0.19072255756326	9.070E-12	-1.8654133E-6
-0.0	0.00000000000000	0.000	0.0000000
0.2	-0.19072255756326	-9.070E-12	1.8654133E-6
0.4	-0.32711140753927	-1.372E-11	2.2509075E-6
0.6	-0.36137118297282	-1.318E-11	1.19149238E-6

0.8	-0.25824819272383	-1.100E-11	1.6904186E-7
1.0	0.00000000000000	-1.052E-11	0.0000000

From the tables 4 and 5 above, one can observe that as the number of terms involved increase, the series solution obtained by differential transform method converges to the series expansion of the exact solution (27).

4. Conclusion

It is shown that the differential transform method can be used successfully for finding the solution of linear and nonlinear boundary value problems of eighth-order. It may be concluded that this technique is very powerful and efficient in finding semi numerical and analytical solutions for these types of boundary value problems.

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