Waves and Ripples in Liquid Films

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الملخـــص

يتضمن هذا البحث وضـــع نمـــوذج رياضـــي للجريـــان الثنـــائي البعـــد
واللامنضغط في غشاء متناظر عندما يكون تأثير قوى اللزوجة صغيراً جداً بالنسبة
إلى قوى الشد السطحي وقوى القصور الذاتي وقد تم الحـــصول علـــى المعـــادلات
التفاضلية التي تحكم هذا الجريان وقد تم ايجاد حل هذه المعادلات، كما تمت دراسة
الموجات في هذا الغشاء.

ABSTRACT

In this paper we present a mathematical model for two-dimensional incompressible flow in a symmetric thin liquid films with the viscosity forces, which can be very small, compared with surface tension and inertia forces. We obtain the governing differential equation for such flow, we also determine the solution of equations and also we consider an inviscid waves in thin films.

1- Introduction:

A problem in fluid mechanics, which received some attention recently, concerns many different types of doubts about the boundary conditions at the surface of liquid films. In the motion of a very thin soap films see for example [5], it has been suggested that an appropriate boundary condition is the kinematic condition of rigidity (inextensible and flexible) rather than the dynamic condition of zero shear stress at the bounding surfaces [1]. Indeed the only existing theory for such motion is a varient of lubrication theory [6], based on this kinematics condition. If, on the other hand, the ordinary zero shear stress condition at the surface is adopted, then the velocity of along the film is almost the same, and this idea is used by [2] when there is a balance between surface tension, inertia and visions forces, and it is also used by [3] for flow in thin liquid films with negligable inertia.

The mechanics of a free surface film flowing steadily between two vertical guide wires was invistigated by [4] and shows that the analytic solution can be applied in a loatig process.

The objective of this paper is to develop the corresponding theory for more conventional dynamic condition of fluid motion within a symmetric film as shown in figure (1.1) whose flow is in the x. direction of the coordinate axes x and y, and to achieve the derivation of the governing equation for thickness of the liquid film when the flow is inviscid, for both steady and unsteady motion. Furthermore, we consider inviscid waves of small amplitude and the periodic solution of the governing differential equation.
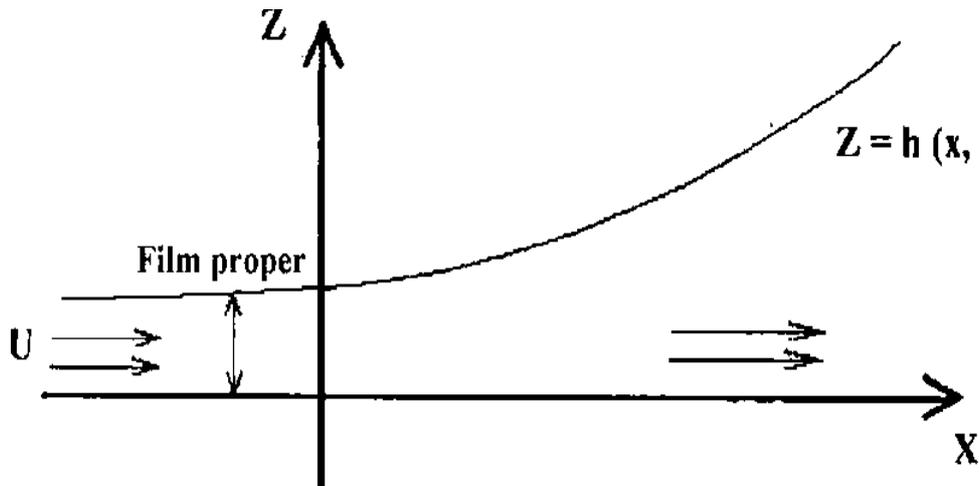
Figure (1.1) Cross section of symmetric film.

2- Governing Differential Equations:

The slope of the film surface is zero on the film proper, and in the transition region remains small approximately in the ratio of its width to a typical value of the radius of curvature, only on the border does the slope reach a substantial values, and thus we take $\partial h/\partial c \ll 1$ over the domain x.

Let $z = \pm h (x, t)$ be the equation of the film surfaces, we assume here a two-dimensional incompressible flow.

Following [2], the governing differential equations of motion of flow within a double sided films are given by:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0, \quad \text{..(2.1)}$$

$$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}) = \sigma\frac{\partial^3 h}{\partial x^3} + 3\mu\frac{\partial^2 u}{\partial x^2}, \text{..(2.2)}$$

Where p is the density of the fluid, p, the dynamic viscosity,σ the surface tension, and u is the component of velocity along the x - axis.

189

3- Inviscid Waves of Small Amplitude:

In the governing differential equations (2.1) and (2.2), if we set

$$h = h_0 \{1 + \varepsilon f(x,t)\}, \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3.1)$$

$$u = \varepsilon g(x,t) , \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3.2)$$

where f, g, f′, g′, f″ and g″ are continuous functions and e << 1, then equations (2.1) and (2.2) give after some manipulation, the following fourth order partial differential equation:

$$\rho \frac{\partial^2 f}{\partial t^2} - \frac{3\mu \partial^3 f}{\partial x^2 \partial t} + \sigma h_o \frac{\partial^4 f}{\partial x^4} = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots (3.3)$$

Equation (3.3) has a fundamental solution of the form

$$f(x, t) = e^{k(ix - \alpha kt)} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3.4)$$

Differentiate (3.4) with respect to x and t, and then substitute these derivatives in equation (3.3), we get

$$\rho\alpha^2 - 3\alpha\mu + \sigma h_o = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3.5)$$

If the viscosity (J, tends to zero for inviscid approximation, then (3.5) gives

$$\rho\alpha^2 + \sigma h_o = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3.6)$$

or

$$\alpha = \pm i (\sigma \frac{h_o}{\rho})^{1/2}$$

The fundamental solution (3.4) can be expressed as a linear combination of solutions, yields

$$f(x, t) = \sin(kx - \alpha k2t)$$

or

$$f(x, t) = \sin[k(x - ct)]\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3.7)$$

where $\alpha = \dfrac{c}{k}$, c is the phase velocity, and k is the wave number.

Now differentiate (3.7) with respect to x and t, then substitute in equation (3.3), we get

$$c = k \, (\frac{\sigma h_o}{\rho})^{1/2} \quad\text{.. (3.8)}$$

A result which is consistent with the usual theory of propagation of surface tension ripples on a liquid of depth ho, and equation (3.8) is comparable with the theory of irrational surface tension waves in a channel of depth ho, namely

$$C = \frac{\sigma k}{\rho} \tanh (kh_o) \quad\text{.. (3.9)}$$

Provided that $kh_o \ll 1$. This is the usual approximation for long-waves in shallow water, not normally relevant to surface tension waves in channel theory, since such waves are dominated by gravity.

In terms of wave - length λ, where

$$\lambda = \frac{2\pi}{k}, \quad\text{.. (3.10)}$$

equations (3.8) and (3.10) give the wave speed c,

$$c = \frac{2\pi}{\lambda}(\tau h_o)^{1/2}, \quad\text{.. (3.11)}$$

and period

$$T = \frac{\lambda}{c} = \frac{\lambda^2}{2\pi(\tau h_o)^{1/2}}, \quad\text{.. (3.12)}$$

Where $\tau = \frac{\sigma}{\rho}$ is the kinematic surface tension, and it is the only material constant in these inviscid flows.

The following data for p and *a* refer to a standard 20C° temperature and atmospheric pressure, for $\lambda=1$ and for the reliable application of continuum mechanics, a liquid film must be at least 100 molecules thick, we take $h_o=0.01$ cm, and then from equations (3.11) and (3.12) we can determine the wave speed c and the period T as shown in the following table.

191

Liquid	Water	Mercury	Glycerin	Carbon tetrachloride	Linseed oil	Olive oil	Turp-entine
m gm/cm.sec	0.0113	0.0155	14.9	0.00974	0.4309	0.8379	0.0149
r gm/cm^3	0.998	13.55	1.26	1.59	0.94	0.91	0.86
σ gm/sec^2	72.97	510.76	62.75	26.27	33.57	33.56	26.27
τ cm^3/sec^2	73.12	37.69	49.80	16.52	35.70	36.88	30.55
c cm/sec	5.37	3.86	4.43	2.55	3.75	3.82	3.47
T sec	0.186	0.259	0.226	0.392	0.266	0.266	0.288

Table (3.1): The wave speed and the period for various liquids.

4- Sinusoidal Wave :

The theory of inviscid irrational surface waves in a channel of depth h_o gives.

$$c^2 = \left(\frac{g}{k} + \frac{\sigma k}{\rho} \right) \text{than} (kh_o) \dots\dots (4.1)$$

Now equations (3.8) and (3.12) with equation (4.1) are comparable if :

a) For short wave limit, the wave length X is very small and kh_o is very large, but $\tanh(kh_o)$ is asymptotic to one as kh_o becomes very large. Thus equation (4.1) gives

$$c = \left[\frac{g}{k} + \frac{\sigma k}{\rho} \right]^{\frac{1}{2}} \dots\dots (4.2)$$

Now if $\lambda_m = L = 2\pi(\dfrac{\sigma}{\rho g})^{1/2}$, ...(4.3)

then

$$\lambda \ll \lambda_m$$

where λ_m denotes the maximum wave length, and so

$$\lambda \ll 2\pi\left(\dfrac{\sigma}{\rho g}\right)^{\frac{1}{2}} \quad \text{...(4.4)}$$

By using equations (3.10) and (4.4), we get

$$\dfrac{\sigma k^2}{\rho g} > 1, \quad \text{...(4.5)}$$

b) For long wave limit, X is very large and thus kh_o is very small, that is

$$k2h_o^2 \ll 1 \text{...(4.6)}$$

But $\tanh(kh_o)$ is asymptotic to kh_o, thus equation (4.1), gives

$$c^2 = (g + \dfrac{\sigma k^2}{\rho})h_o, \quad \text{...(4.7)}$$

if we neglect the effect of gravity, then equation (4.7) gives

$$c = k(\dfrac{\sigma h_o}{\rho})^{1/2} \quad \text{...(4.8)}$$

Which is the same as equation (3.12), such waves in which the only significant restoring force is surface tension, the force responsible for capillary attraction" are often called capillary waves. In water the capillary waves are waves with $\lambda < 0.4$ cm, so that it is easy to excite them by striking a tuning fork and placing the tines in the water, see [5]. For example, the tuning fork generates by equation (4.8) wave length $\lambda \cong 0.171$ cm, with velocity c = 31.42 cm/sec.

The short and long wave conditions (4.5) and (4.6) respectively are not incomparable for some liquids.

193

The value of L from equation (4.3) for water is (1.71 cm) and for mercury is (1.23 cm), other values of L can be determined from equation (4.3), more further, the conditions in term of the wave length λ, respectively becomes:

a) $\dfrac{\lambda^2}{L^2} << 1$ ....(4.9)

b) $\dfrac{\lambda^2}{L^2} << \left(\dfrac{4\pi h_o}{L} \right)$ ....(4.10)

c) $\dfrac{h_o}{H} >> 1,$ ....(4.11)

Where $H = \dfrac{9\mu^2}{4\sigma\rho}$ ....(4.12)

From table (3.1), it is obvious that the inviscid approximation is relevant water and mercury since the viscosity p, is very small compared with the other liquids in which this approximation is not relevant to them. The following figure represents the relation between the maximum wave length λ_m and the wave speed c for water and mercury.
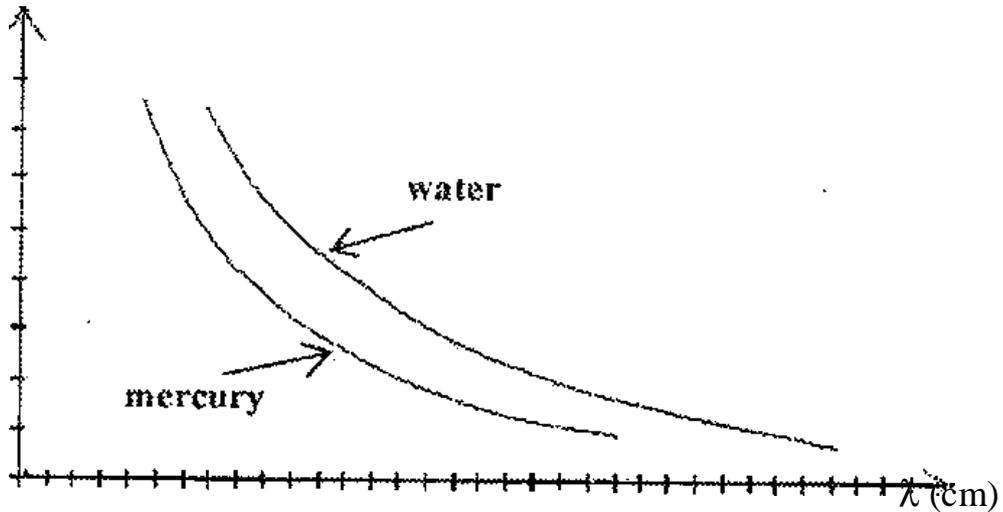
194

c (cm/sec)

Figure (4.1): The wave length l and the wave speed c
for
water and mercury.

From equations (4.3), (4.12), we can determine L and H for
water and mercury, which are

$$H_{water} = 3.94 \times 10^{-6}, \qquad L_{water} = 1.71 \text{ cm}$$
$$H_{marcury} = 7.8 \times 10^{-8}, \qquad L_{mercury} = 1.23 \text{ cm}$$

The following table gives some values for λ and h_o.

Water		Mercury		Water		Mercury	
λ	h_o	λ	h_o	$\log\left(\dfrac{\lambda}{L}\right)$	$\log\left(\dfrac{h_o}{H}\right)$	$\log\left(\dfrac{\lambda}{L}\right)$	$\log\left(\dfrac{h_o}{H}\right)$

195

2.5×10^{-3}	3.94×10^{-4}	4.5×10^{-5}	7.8×10^{-6}	2.8	2	4.4	2
10^{-2}	8×10^{-3}	10^{-4}	1.6×10^{-6}	2.23	2.6	4.1	2.3
5×10^{-2}	1.6×10^{-3}	10^{-3}	1.6×10^{-4}	1.53	3.3	3.1	3.3
10^{-1}	1.6×10^{-2}	10^{-2}	1.6×10^{-3}	1.23	3.6	2.1	4.3
1.71×10^{-1}	2.7×10^{-2}	1.23×10^{-1}	1.9×10^{-2}	.1	3.8	.1	45

Table (4.1) values for l and ho for various liquids.

Now the limit of one percentage error in each approximation gives the following domains of validity of all three approximations in (h_o, λ) plane as shown in diagram (4.1).
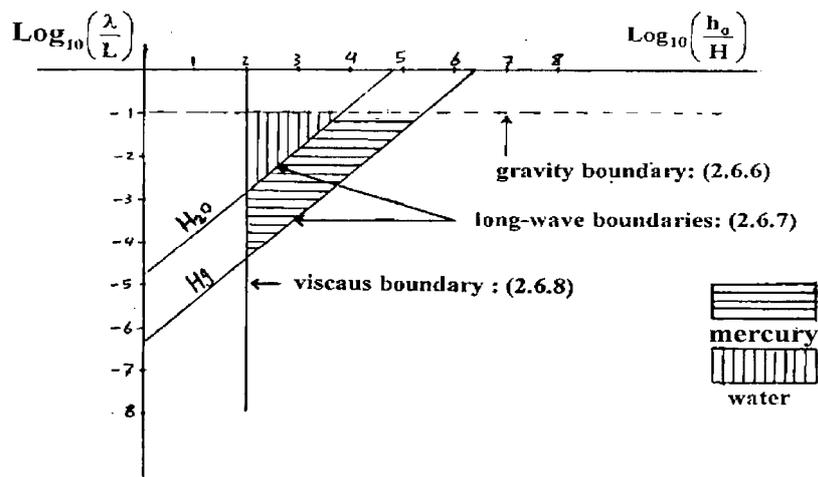
The separate limits of 1% error in h_o, λ are:

	H_o (cm)		l (cm)	
	Minimum	Maximum	Minimum	Maximum
Hg	9.01×10^{-6}	1.88×10^{-9}	5.65×10^{-4}	1.18×10^{-1}

196

H$_2$O	3.91 × 10^{-4}	2.77 × 10^{-3}	2.43 × 10^{-2}	1.72 × 10^{-1}

Diagram (4.1): Domains of validity for approximation in (h$_o$, l) plane.

5- Conclusion:

The approximation of inviscid waves of small amplitude shows that it is a usual approximation for long waves in shallow water and normally relevant to surface-waves in channel theory since such waves are dominated by gravity. The **wave** speed c and the period T are evaluated for some liquids, namely water and mercury.

The domain of validity of all approximations in the case of sinusoidal waves when h$_o$ >> H are investigated, and in this case the only a symptomatic limit h$_o$ >> H → ∞ is relevant and this limit may be regarded as the limit H → 0, which means that we consider the effect of a very small but non-zero value of viscosity μ.

REFERENCES

[1]   Abdulahad. J.G., (1989), Liquid surface conditions, Ph.D. thesis. University of Essex.

[2]   Abdulahad. J.G., (1994), Free films in homogenous liquids, J. Educ. Sci. 268,17.

[3]   Abdulahad. J.G., (1994), Fluid flow in thin liquid films with negligable inertia. J. Educ. Sci. 278c, 17.

[4]   Cyrus K.A., (1987), Mechanics of a free surface liquid Film flow. J. Applied mechanics. Vol. 54.

[5]   Mysels, K.J. Shinoda, K. and Franki, S., (1959), Soap films, studies of their thining and a biblography, pergamon press. London.

[6]   Pinkus. O.S., and Stermlicht, (1961), Theory of Hydrodynamic lubrication. McGraw Hill. New York.

198