

Stability Analysis of Unified Chaotic System

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Received on: 8/3/2009

Accepted on: 17/5/2009

ABSTRACT

The aim of this paper is to find the stability of unified chaotic system through studying different systems. The unified chaotic system is divided into three systems by depending on the values of θ , when $0 \leq \theta < 0.8$, the unified chaotic system becomes Lorenz system; when $\theta = 0.8$; it becomes Lü system, when $0.8 < \theta \leq 1$; the unified chaotic system becomes Chen system. Investigations the stability analysis of these systems leads to the stability of the unified chaotic system.

Keywords: unified chaotic system, Lorenz family, stability, Routh-Hurwitz

تحليل استقرارية النظام المضطرب الموحد

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تاريخ القبول: 2009/05/17

تاريخ الاستلام: 2009/03/08

المخلص

الهدف من هذا البحث هو إيجاد استقرارية النظام المضطرب الموحد من خلال دراسة أنظمة مختلفة, حيث ينقسم النظام الموحد إلى ثلاثة أنظمة بالاعتماد على قيم θ , بحيث عندما تكون $0 \leq \theta < 0.8$ فان النظام المضطرب الموحد يصبح نظام لورنس وعندما $\theta = 0.8$ فان النظام الموحد يصبح نظام لي وعندما $0.8 < \theta \leq 1$ فان النظام الموحد يصبح نظام تشين . وبدراسة تحليل استقرارية هذه الأنظمة يقودنا إلى استقرارية النظام المضطرب الموحد.

الكلمات المفتاحية: النظام المضطرب الموحد, عائلة لورانس, الاستقرارية, معايير-Routh-Hurwitz

1- Introduction:

Over the last three decades, Chaos as very interesting nonlinear phenomena has been widely studied in various scientific fields. Recently, there has been increasing interest in controlling and utilizing chaos, particularly within the physicists and mathematicians and engineering and technological communities [5].

In 1963, Lorenz found the first chaotic system, which is a third order autonomous system with only two multiplication-type quadratic terms but displays very complex dynamical behaviors [10]. In 1999, Chen found another similar but topologically non-equivalent chaotic system, Chen system[3]. In 2002, Lü and Chen found a new chaotic system, bearing the name of the Lü system [1]. In the same year, Lü et al. unified above the three chaotic systems into one chaotic system which is called unified chaotic system [2].

The unified chaotic system was produced as a kind of unique and unified classification between the Lorenz and the Chen attractors, both in theory and in simulation. Of particularly interesting is that this unified system is likely the simplest chaotic system that bridges the gap between the Lorenz and the Chen system, and so it contributes to a better understanding of a family of 3-dimensional autonomous quadratic chaotic systems [4].

The nonlinear differential equations that describe the unified chaotic system are following form [2, 4, 8]:

$$\left. \begin{aligned} \dot{x} &= (25\theta + 10)(y - x) \\ \dot{y} &= (28 - 35\theta)x - xz + (29\theta - 1)y \\ \dot{z} &= xy - \frac{\theta + 8}{3}z \end{aligned} \right\} \dots (1)$$

Where $\theta \in [0, 1]$, this system which describes a large family of chaotic systems containing the Lorenz and Chen system as two extremes and the Lü system as a transition in between[10]. Recently, there are some analytical results reported about these chaotic systems, which are called the Lorenz family [3].

When $\theta \in [0, 0.8)$, system (1) is called the generalized Lorenz system; when $\theta = 0.8$; it becomes the generalized Lü system; when $\theta \in (0.8, 1]$; system (1) is called the generalized Chen system [2, 9].

The three critical points of system (1) can be found in [4] as follows:

$$\begin{aligned} S_0 & (0, 0, 0) \\ S_- & (-\sqrt{(8+\sigma)(9-2\sigma)}, -\sqrt{(8+\sigma)(9-2\sigma)}, 27-6\sigma) \\ S_+ & (\sqrt{(8+\sigma)(9-2\sigma)}, \sqrt{(8+\sigma)(9-2\sigma)}, 27-6\sigma), \end{aligned}$$

and studied the stability at these points. Finally, cleared that this system is unstable for $\theta \in [0, 1]$, without depended on the Lorenz, Chen and Lü systems.

In this paper, we find the stability conditions of Lü system and obtain on helping results; finally, we find the same results which found in [4], but with

another method, by using the three chaotic systems to investigate the stability of system (1).

2- Preliminaries:

The following notations will be used for the remaining of this paper.

The mathematical model of **Lorenz system** is a system of nonlinear ordinary differential equations which has the following form:

$$\left. \begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - xz - y \\ \dot{z} &= xy - \beta z \end{aligned} \right\} \dots(2)$$

where $\sigma, r, \beta > 0$, and Lorenz studied this system when $\sigma = 10, r = 28, \beta = 8/3$, [6].

Chen system it also has the form of nonlinear ordinary differential equations:

$$\left. \begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= (r-\sigma)x - xz + ry \\ \dot{z} &= xy - \beta z \end{aligned} \right\} \dots (3)$$

where $\sigma, r, \beta > 0$ and $\sigma > r$, Chen take $\sigma = 35, r = 28, \beta = 3$ [5].

The mathematical model of **Lü system** also is a system of nonlinear ordinary differential equations which has the form:

$$\left. \begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= -xz + ry \\ \dot{z} &= xy - \beta z \end{aligned} \right\} \dots(4)$$

also $\sigma, r, \beta > 0$, $\sigma > r$, Lü studied this system when $\sigma = 36, r = 20, \beta = 3$ [1].

One can see that the three dynamical systems, all are nonlinear ordinary differential equation systems of third degree. And all x derivative in those systems are multiplied by a number σ . The only difference is the form of y derivative [1]. According to Vanecek and Celikovsky definition. They separated the system into linear and quadratic parts. In the linear part of system described by the matrix $A = (a_{ij})_{3 \times 3}$, the Lorenz system satisfies the condition $a_{12} a_{21} > 0$ while the Chen system satisfies $a_{12} a_{21} < 0$ and Lü system satisfies the condition $a_{12} a_{21} = 0$ [8, 9].

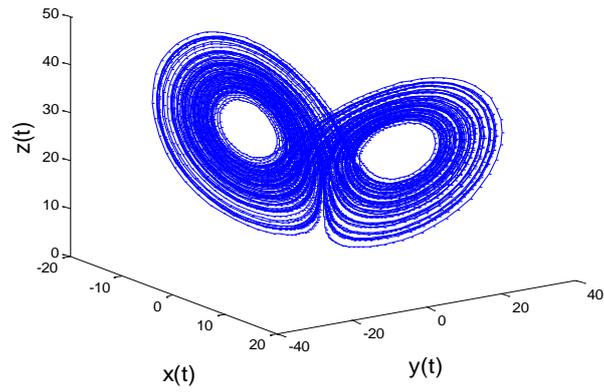


Figure 1: the attractor of Lorenz system

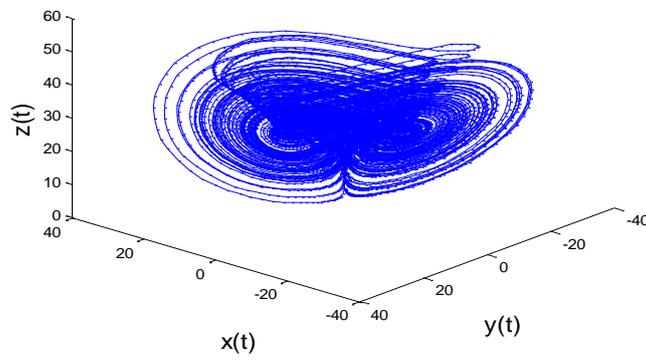


Figure 2: the attractor of Lü system

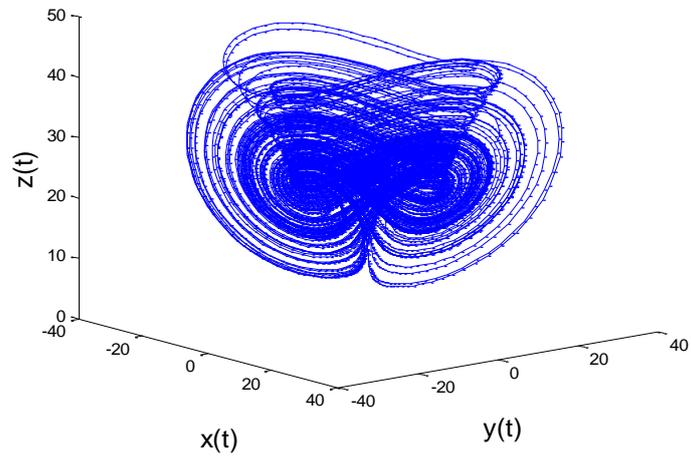


Figure 3: the attractor of Chen system

3- Helping Results:

Regarding the basic dynamical behaviors of the **Lorenz system**, we have the following observations.

Lemma 1 [7]: In system (2)

(i) if $r < 1$, then this system has only one critical point, $S_0(0, 0, 0)$;

(ii) if $r > 1$, then this system has three critical points : $S_0(0, 0, 0)$

$S_+(x_0, y_0, z_0), S_-(-x_0, -y_0, z_0)$, Where $x_0 = y_0 = \sqrt{\beta(r-1)}$ and

$$z_0 = r - 1.$$

In fact, linearized system (2) about the critical point $S_0(0, 0, 0)$ yields the following characteristic equation [7]:

$$f(\lambda) = \lambda^3 + (\sigma + 1 + \beta)\lambda^2 + [\beta(\sigma + 1) + \sigma(1 - r)]\lambda + \beta\sigma(1 - r) = 0 \quad \dots(5)$$

While the linearized system (2) about the critical point S_+ or S_- yields the following characteristic equation [7]:

$$f(\lambda) = \lambda^3 + (\sigma + 1 + \beta)\lambda^2 + \beta(\sigma + r)\lambda + 2\beta\sigma(r - 1) = 0 \quad \dots(6)$$

Theorem 1 [8]: In system (2)

(i) if $r < 1$, then the critical point $S_0(0, 0, 0)$ is asymptotically stable ;

(ii) if $r > 1$, then the critical point $S_0(0, 0, 0)$ is unstable ;

(iii) if $r < r_c = \frac{\sigma(\sigma + \beta + 3)}{\sigma - \beta - 1}$, then the critical points S_+ and S_- are all asymptotically stable ;

(iii) if $r > r_c$, then the critical points S_+ and S_- are all unstable .

Now, the basic dynamical behaviors of the **Chen system** is included in the following lemma.

Lemma 2 [5]: In system (3)

(i) if $\sigma > 2r$, then this system has only one critical point, $S_0(0, 0, 0)$;

(ii) if $\sigma < 2r$, then this system has three critical points : $S_0(0, 0, 0)$

$S_+(x_0, y_0, z_0), S_-(-x_0, -y_0, z_0)$, where $x_0 = y_0 = \sqrt{\beta(2r - \sigma)}$ and

$$z_0 = 2r - \sigma.$$

Linearized system (3) about the critical points S_0 and S_+ or S_- yields the following characteristic equations respectively [5]:

$$f(\lambda) = \lambda^3 + (\sigma + \beta - r)\lambda^2 + [\beta(\sigma - r) + \sigma(\sigma - 2r)]\lambda + \beta\sigma(\sigma - 2r) = 0 \quad \dots(7)$$

$$f(\lambda) = \lambda^3 + (\sigma + \beta - r)\lambda^2 + \beta r\lambda + 2\beta\sigma(2r - \sigma) = 0 \quad \dots(8)$$

Theorem 2 [5]: In system (3)

(i) if $\sigma > 2r$, then the critical point $S_0(0, 0, 0)$ is asymptotically stable;

(ii) if $\sigma < 2r$, then the critical point $S_0(0, 0, 0)$ is unstable ;

(iii) if $\sigma < 2r$ and $(\sigma + \beta - r)r - 2\sigma(2r - \sigma) > 0$, then the critical points s_+ and s_- are all stable .

4- Main results:

It is easy to verify that the $L\ddot{u}$ system has three critical points: $S_0(0, 0, 0)$, $S_+(x_0, y_0, z_0)$, $S_-(-x_0, -y_0, z_0)$, where $x_0 = y_0 = \sqrt{\beta r}$ and $z_0 = r$

Proposition 1: The critical point $S_0(0, 0, 0)$ is always unstable.

Proof: At the critical point $S_0(0, 0, 0)$, is linearized, the Jacobian matrix of system (4) is defined as ;

$$J_{S_0} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 0 & r & 0 \\ 0 & 0 & -\beta \end{bmatrix} \quad \dots(9)$$

The characteristic equation is:

$$f(\lambda) = \lambda^3 + (\sigma + \beta - r)\lambda^2 + [\beta(\sigma - r) - \sigma r]\lambda - \beta\sigma r = 0 \quad \dots(10)$$

$$\text{Or } (\lambda + \beta) \cdot (\lambda^2 + (\sigma - r)\lambda - \sigma r) = 0 \quad \dots(11)$$

either $(\lambda + \beta) = 0$ or $\lambda^2 + (\sigma - r)\lambda - \sigma r = 0$, and three eigenvalues corresponding to the critical point S_0 are:

$$\lambda_1 = -\beta, \quad \lambda_2 = -\sigma, \quad \lambda_3 = r$$

So, it's clear that the parameter $r > 0$, therefore $\lambda_3 > 0$, consequently the critical point $S_0(0, 0, 0)$ is always unstable. This completes the proof.

In the following, we consider the stability of the system (4) at the critical points s_+ and s_- , Because the system is invariant under the transformation[1], so one only needs to consider the stability of any one of the both . The stability of the system (4) at critical point s_+ is analyzed in this paper.

Under the linear transformation $(x, y, z) \rightarrow (X, Y, Z)$:

$$\left. \begin{aligned} x &= X + \sqrt{\beta r} \\ y &= Y + \sqrt{\beta r} \\ z &= Z + r \end{aligned} \right\} \quad \dots(12)$$

the system (4) becomes

$$\left. \begin{aligned} \dot{X} &= \sigma(Y - X) \\ \dot{Y} &= -rX + rY - \sqrt{\beta r} Z \\ \dot{Z} &= \sqrt{\beta r} X + \sqrt{\beta r} Y - \beta Z \end{aligned} \right\} \dots(13)$$

The critical point s_+ of the system (4) is switched to the new critical point $S'_0(0, 0, 0)$ of the system (13) under the linear transformation, in the following, the stability of system (13) at the critical point S'_0 is considered.

The Jacobian matrix of the system (13) at $S'_0(0, 0, 0)$ is :

$$J(S'_0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ -r & r & -\sqrt{\beta r} \\ \sqrt{\beta r} & \sqrt{\beta r} & -\beta \end{bmatrix} \dots(14)$$

and the characteristic equation is :

$$f(\lambda) = \lambda^3 + (\sigma + \beta - r)\lambda^2 + \beta\sigma\lambda + 2\beta\sigma r = 0. \dots (15)$$

Theorem 3: In the system (4)

- (i) if $\sigma + \beta > 3r$, then the critical point s_+ is asymptotically stable ;
- (ii) if $\sigma + \beta < 3r$, then the critical point s_+ is unstable .

Proof: From the equation (15), Let

$$\left. \begin{aligned} A &= \sigma + \beta - r \\ B &= \beta\sigma \\ C &= 2\beta\sigma r \end{aligned} \right\} \dots (16)$$

Then the Routh-Hurwitz conditions lead to the conclusion that the real parts of the roots are negative if and only if $A > 0, C > 0$ and $AB > C$.

$\sigma + \beta - r > 0$ always, since $\sigma > r$ and clear that $2\beta\sigma r > 0$, consequently we must prove that $(\sigma + \beta - r)\beta\sigma > 2\beta\sigma r$

$$(\sigma + \beta - r)\beta\sigma > 2\beta\sigma r \Rightarrow \sigma + \beta - r > 2r$$

\therefore when $\sigma + \beta > 3r$ Then the critical point s_+ is asymptotically stable and when $\sigma + \beta < 3r$ then the critical point s_+ is unstable. This completes the proof.

Now, to simplify computations for our results, we described the system (1) as follows:

$$\left. \begin{aligned} \dot{x} &= \sigma_1(y - x) \\ \dot{y} &= r_1x - xz + r_2y \\ \dot{z} &= xy - \beta_1 z \end{aligned} \right\} \dots(17)$$

Where $\sigma_1 = 25\theta + 10, r_1 = 28 - 35\theta, r_2 = 29\theta - 1, \beta_1 = \frac{\theta + 8}{3}$.

The system (17) has three critical points which are $S_0(0, 0, 0)$, $S_+(x_0, y_0, z_0)$, $S_-(-x_0, -y_0, z_0)$, where $x_0 = y_0 = \sqrt{\beta(r_1 + r_2)}$ and $z_0 = r_1 + r_2$.

The Jacobian matrix of system (17) at the critical point S_0 is defined as:

$$J_{S_0} = \begin{bmatrix} -\sigma_1 & \sigma_1 & 0 \\ r_1 & r_2 & 0 \\ 0 & 0 & -\beta_1 \end{bmatrix} \quad \dots(18)$$

and the characteristic equation is

$$f(\lambda) = \lambda^3 + (\sigma_1 + \beta_1 - r_2)\lambda^2 + [\beta_1(\sigma_1 - r_2) - \sigma_1(r_1 + r_2)]\lambda + \beta_1\sigma_1(-r_1 - r_2) = 0 \quad \dots(19)$$

We will use the following Corollary, which enables us to find the Jacobian matrix directly for Lorenz family at S_+ and S_- without using transformation.

Corollary1:

Let $D = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$ is a Jacobian matrix at S_0 , then

$E = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$ is a Jacobian matrix at S_+ ,

where $e_{11} = d_{11}$, $e_{21} = -d_{22}$, $e_{31} = x_0$,
 $e_{12} = d_{12}$, and $e_{22} = d_{22}$, and $e_{32} = x_0$.
 $e_{13} = d_{13}$, $e_{23} = -x_0$, $e_{33} = d_{33}$

So, the Jacobian matrix at S_+ by using Corollary 1 is given by:

$$J(S'_0) = \begin{bmatrix} -\sigma_1 & \sigma_1 & 0 \\ -r_2 & r_2 & -\sqrt{\beta_1(r_1 + r_2)} \\ \sqrt{\beta_1(r_1 + r_2)} & \sqrt{\beta_1(r_1 + r_2)} & -\beta_1 \end{bmatrix} \quad \dots (20)$$

and the characteristic equation is :

$$f(\lambda) = \lambda^3 + (\sigma_1 + \beta_1 - r_2)\lambda^2 + \beta_1(\sigma_1 + r_1)\lambda + 2\beta_1\sigma_1(r_1 + r_2) = 0 \quad \dots(21)$$

Corollary2:

The third coefficient values (C) of the cubic equations (5,6,7,8,10,15,19,21) depended on the third value of S_+ (z_0) as following :

$$C_5 = C_7 = C_{10} = C_{19} = -\beta\sigma(z_0)$$

and $C_6 = C_8 = C_{15} = C_{21} = 2\beta\sigma(z_0)$.

Theorem 4: In system (17)

(i) The critical point S_0 is unstable if one of the following cases holds:

- 1 $0 \leq \theta < 0.8$ and $r_1 > 1$;
- 2 $\theta = 0.8$ for any value of $\sigma_1, r_1, r_2, \beta_1$;
- 3 $0.8 < \theta \leq 1$ and $\sigma_1 < 2r_1$.

(ii) The critical point S_+ is unstable if one of the following cases holds:

- 1 $0 \leq \theta < 0.8$ and $r_1 > r_c = \frac{\sigma_1(-\sigma_1 - \beta_1 + 3r_1)}{-\sigma_1 + \beta_1 - r_2}$;
- 2 $\theta = 0.8$ and $\beta_1 + \sigma_1 < 3r_1$;
- 3 $0.8 < \theta \leq 1$ for any value of $\sigma_1, r_1, r_2, \beta_1$,

Proof:

(i)(1) when $\theta \in [0, 0.8)$, the system (17) is the generalized system(2), and $r_1 \cong r$, since $r_1 > 1$ (given), then $r > 1$ also , the system (2) is unstable (Theorem 1, case(ii)) , therefore system (17) is unstable at S_0 .

(2) when $\theta = 0.8$,system (17) becomes the generalized system(4),and by Proposition 1,then system(4) is always unstable, consequently system (17) is unstable at S_0 .

(3) when $\theta \in (0.8, 1]$, system (17) belongs to the generalized system(3),and $\sigma_1 \cong \sigma$, $r_1 \cong r$, since $\sigma_1 < 2r_1$ (given) ,then $\sigma < 2r$ and satisfied theorem 2 case (ii) ,so system(3) is unstable, therefore system (17) is unstable also at S_0 .

(ii)(1) when $\theta \in [0, 0.8)$, system (17) is the generalized system(2), and $r_1 \cong r$, since $r_1 > r_c = \frac{\sigma_1(-\sigma_1 - \beta_1 + 3r_1)}{-\sigma_1 + \beta_1 - r_2}$ (given) ,then $r > \frac{\sigma(\sigma + \beta + 3)}{\sigma - \beta - 1}$ also , so system (2) is unstable (Theorem 1 ,case(iiii)) , therefore system (17) is unstable at S_+ .

(2) when $\theta = 0.8$,system (17) becomes the generalized system(4), and $\sigma_1 \cong \sigma$, $r_1 \cong r$, $\beta_1 \cong \beta$, and $\beta_1 + \sigma_1 < 3r_1$ (given) then $\beta + \sigma < 3r$, so satisfied theorem 3 case (ii) ,hence system(4) is unstable, consequently system (17) is unstable also at S_+ .

(3) when $\theta \in (0.8, 1]$, system (17) belongs to the generalized system(3),and from equation 8, we have $A > 0$, $C > 0$ but $AB < C$, not satisfy Routh-Hurwitz criterion hence the real part of at least one root is positive , therefore system(3) is unstable, consequently system (17) is unstable also at S_+ .

5 - Illustrative Example:

Example: Investigate for stability of system 1

Solution: if $\theta = 0$, then system (1) becomes system(2)

$$\left. \begin{aligned} \dot{x} &= 10(y - x) \\ \dot{y} &= 28x - xz - y \\ \dot{z} &= xy - \frac{8}{3}z \end{aligned} \right\} \dots (22)$$

By theorem (4), case (i) first condition, the system (22) is unstable at S_0 since $r = 28 > 1$ and unstable at S_+ since $r = 28$ and $r_c = -85.6$ so $r > r_c$.

if $\theta = 1$, then system (1) becomes system (3)

$$\left. \begin{aligned} \dot{x} &= 35(y - x) \\ \dot{y} &= (28 - 35)x - xz + 28y \\ \dot{z} &= xy - 3z \end{aligned} \right\} \dots (23)$$

Since $\sigma = 35$, $2r = 56$ hence $\sigma < 2r$ therefore satisfied theorem (4), case (i) third condition therefore system (23) is unstable at S_0 and also unstable at S_+ by the same theorem, case (ii) third condition.

if $\theta = 0.8$, then system (1) becomes system (4)

$$\left. \begin{aligned} \dot{x} &= 30(y - x) \\ \dot{y} &= -xz + 23.2y \\ \dot{z} &= xy - \frac{8.8}{3}z \end{aligned} \right\} \dots (24)$$

By theorem (4), case (i) second condition, the system (24) is unstable at S_0 and unstable at S_+ since $\beta + \sigma = 32.93$ while $3r = 69.9$, hence $\beta + \sigma < 3r$ therefore satisfied theorem (4), case (ii) third condition.

6 - Conclusion:

In this paper, we have investigated the stability of unified chaotic system by using three chaotic systems (Lorenz, Chen and Lü systems). By this method we justified the same results which have been found by previous methods. An illustrative example shows the effectiveness and feasibility of this method.

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