# On SNF-rings , I

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**ABSTRACT**

A ring $R$ is called right SNF-rings if every simple right R-module is N-flat . In this paper , we give some conditions which are sufficient or equivalent for a right SNF-ring to be n-regular (reduced) .It is shown that

1- If $r(a)$ is a GW-ideal of R for every $a \in R$. then , $R$ is reduced if and only if $R$ is right SNF-ring.

2- If $R$ is an reversible, then $R$ is regular if and only if $R$ is right GQ-injective and SSNF-ring .

Key words: SNF-rings, GW-ideal ,reversible.

**حول الحلقات من النمط SNF ,I**

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**الملخص**

يقال للحلقة R حلقة يمنى من النمطSNF, إذا كان كل مقاس أيمن بسيط على الحلقة R مسطحاً من النمط N . في هذا البحث نحن أعطينا بعض الشروط التي تحقق أو تكافئ الحلقات اليمنى من النمط SNF ، حلقات منتظمة من النمط n (مختزلة) . تبين لنا

1- إذا كانت $r(a)$مثالي من النمط-GW في الحلقة $R$ لكل $a \in R$ فأن $R$ حلقة مختزلة إذا وفقط إذا كانت $R$ حلقة من النمط-SNF .

2- إذا كانت R حلقة عكوسة فأن Rحلقة منتظمة إذا وفقط إذا كانت حلقة غامرة من النمط GQاليمنى و SSNF.

الكلمات المفتاحية: حلقات من النمط SNF , مثاليات من النمط GW ,حلقة عكوسة,

## 1. Introduction

Throughout this paper, $R$ denotes an associative ring with identity and all modules are unitary .We write $J = J(R)$ for the Jacobson radical of $R$ , and $Y = Y(R)$ ( $Z = Z(R)$) for the right (left) singular ideal of $R$. The right and left annihilators of a subset $X$ of a ring $R$ are written as $r(X)$ and $l(X)$. A right R-module M is said to be flat if , given any monomorphism $N \to Q$ of left R-modules $N$ and $Q$, the induced homomorphism $M \otimes N \to M \otimes Q$ is also monomorphism [1] . Generalizations on right flat modules have been studied by many authors (see [9] and[3]) . In [5] SF rings are defined and studied . A ring $R$ is called right (left) SF-ring if every simple right (left) R-module is flat . In [9] , Wei and Chen first introduced and characterized a right N-flat modules , and gave many properties . A right R-module is called N-flat , if for any $a \in N(R)$, the map $I_M \otimes i : M \otimes Ra \to M \otimes R$ is monic , where $i : Ra \to R$ is the inclusion mapping . Actually , many authors investigated some properties of rings whose every simple right R-module is N-flat [4] and [9] .

Recall that a ring $R$ is called reduced ring if it has no non zero nilpotent elements , or equivalently , $a^2 = 0$, that implies $a = 0$ for all $a \in R$ . A ring $R$ is called reversible

[2] if for $a,b \in R$, $ab = 0$ implies $ba = 0$. A ring $R$ is said to be Von Neumann regular (or just regular), if $a \in aRa$ for every $a \in R$[5] , a ring $R$ is called n-regular [6] if $a \in aRa$ for all $a \in N(R)$. Clearly, Von Neumann regular ring is n-regular , but the converse is not true by [6, Remark 2.19] .A ring $R$ is said to be right NPP if $aR$ is projective for all $a \in N(R)$ [6] . A right R-module M is called nil-injective if for any $a \in N(R)$, any R-homomorphism $R \to M$ can be extended to $R \to M$ . Or equivalently there exists $m \in M$ such that $f(x) = mx$ for all $x \in aR$ [6,7] . Clearly , a reduced ring is right nil-injective , right NPP and n-regular ring [6] .

## 2. SNF-ring

Following [9] , A ring $R$ is called right (left) SNF if every simple right (left) R-module is N-flat .

The following lemma , which is duo to [9] , plays a central role in several of our proofs .

**Lemma 2.1 :**

1- Let $B$ be a right R-module and there exists R-short exact sequence $0 \to K \overset{j}{\to} F \overset{g}{\to} B \to 0$ where $F$ is N-flat , then $B$ is N-flat if and only if $K \bigcap Fa = Ka$ for all $a \in N(R)$.

2- Let $I$ be a right ideal of $R$ . then $R/I$ is N-flat right R-module if and only if $Ia = I \cap Ra$ for all $a \in N(R)$.

3- Let $R$ be a ring then, $R$ is n-regular ring if and only if every right R- module is N-flat .

Following [5] , a ring $R$ is called MERT ring if every maximal essential right ideal is a two-sided ideal of R.

Clearly , a right SF-ring is right SNF-ring , but the converse is not true. Because there exists a reduced MERT ring which is not regular, there exists a reduced MERT ring $R$ which is not right SF by [12,Theorem 1] . On the other hand ,by [9, Theorem 4.7], reduced ring is right SNF,so there exists a right SNF-ring which is not right SF [9].

**Examples (3) :**

1- Let $Z_2$ be the ring of integer modulo 2 and let $G = \{g : g^3 = 1\}$ be acyclic group ,the group ring $Z_2G = \{0,1, g, g^2, 1+g, 1+g^2, g+g^2, 1+g+g^2\}$ is reduced n-regular ring and SF-ring ,therefore it is SNF-ring .

2- Let $Z_2$ be the ring of integer modulo 2 ,then $R = \left\{\begin{bmatrix} Z_2 & Z_2 \\ Z_2 & Z_2 \end{bmatrix}\right\}$ isSNF-ring but not reduced .

3- The ring of integers $Z$ is SNF-ring but not SF-ring .

**Lemma 2.2: [2]**

Let $R$ be a reversible ring , then $r(a) = l(a)$ for all $a \in R$ .

Following [7], a ring $R$ is said to be right (left) Nduo if $aR$( $Ra$ ) is an ideal of $R$ for all $a \in N(R)$.

**Proposition 2.3 :**

Let $R$ be a right N duo, SNF- ring , then $Y(R)=0$.

**Proof :**

Suppose that $Y(R)\neq 0$ then, $Y(R)$ contains a non-zero element $a$ such that $a^2=0$ .let $x\in l(a)$, $r\in R$. Since $R$ is right N duo , $aR$ is an ideal of $R$. Hence, $ra=at$ for some $t\in R$. Therefore, $xra=xat=0$. This proves that $l(a)$ is a right ideal of $R$. Therefore , there exists a maximal right ideal of $R$ such that $l(a)\subseteq M$ . Since $R$ is right SNF-ring and $a\in l(a)\subseteq M$ , by lemma (2.1), there exists $b\in M$ such that $a=ba$, that is $(1-b)\in l(a)\subseteq M$ and so $1\in M$ , a contradiction . Therefore , $Y(R)=0$ . ■

**Theorem 2.4:**

Let $R$ be a reversible ring .Then , $R$ is a right SNF-ring if and only if $R$ is n-regular ring .

**Proof :**

Suppose that $R$ is n-regular , then $R$ is SNF-ring ,lemma(2.1(3))
Conversely: Let $a\in N(R)$. We claim that $aR+r(a)=R$. If not , then there exists a maximal right ideal $M$ of $R$ such that $aR+r(a)\subseteq M$. Since $R$ is right SNF-ring , $R/M$ is an N-flat right R-module . By lemma (2.1) , $a=xa$ for some $x\in M$ . Since $R$ is reversible , $a=ax$. Hence , $(1-x)\in r(a)\subseteq M$ and so $1\in M$ , which is a contradiction. Therefore , $aR+r(a)=R$ .Hence, $ab+z=1$ for some $b\in R$ and $z\in r(a)$. Since , $az=0$ ,this gives $a=aba$.Thus, $R$ is n-regular.■

From Theorem (2.4) and definition of $C(R)$ we give the following Corollary :

**Corollary 2.5 :**

The center $(C(R))$ of any right (left) SNF-ring is n-regular ring.

Following [13], a left (right) ideal $L$ of a ring $R$ is called generalized weak ideal (GW-ideal), if for any $a\in L$, there existsn>0 such that $a^nR\subseteq L$ ($Ra^n\subseteq L$).

**Theorem 2.6 :**

Let $R$ be a ring such that $r(a)$ is a GW-ideal of $R$ for every $a\in R$. Then , $R$ is reduced if and only if $R$ is right SNF-ring .

**Proof :**

Suppose $R$ is reduced , then $R$ is SNF-ring [9, Theorem 4.2] .
Conversely : Assume that $R$ is SNF-ring and $0\neq b\in R$ such that $b^2=0$. Let $x\in l(b)$, then $b\in r(x)$ . Since r(x) is a GW-ideal of R and $b^2=0$ we have $Rb\subseteq r(x)$. This proves that $l(b)$ is a right ideal of $R$. Therefore , there exists a maximal right ideal $M$ of $R$ such that $l(b)\subseteq M$ . Since $R$ is a right SNF-ring and $b\in l(b)\subseteq M$ by Lemma (2.1) , $b=cb$ for some $c\in M$ , $1-c\in l(b)\subseteq M$ and so $1\in M$, a contradiction , Therefore , $R$ is reduced . ■

Following [8] ,a ring $R$ is called weakly normal if for all $a,r\in R$ and $e\in E(R)$, $ea=0$ implies $areR$ is nil right ideal of $R$ ,where $E(R)$ stands for the set of all idempotent elements of $R$ .

The following result is given in [8]

**Lemma 2.7:**

Let $R$ be a weakly normal ring and $x \in R$.If $x$ is Von Neummanregular ,then $x \in Rx^2 \cap x^2R$.

In the next result, we give another condition for SNF-ring to be a reduced ring .

**Theorem 2.8 :**

Let $R$ be a ring and every principal right ideal is a maximal .Then $R$ is reduced if and only if $R$ is right SNF-ring and weakly normal .

**Proof :**

Let $R$ be reduced, then it is clear $R$ is weakly normal ,and SNF-ring .
Conversely : Let $a \in R$ with $a^2 = 0$ and every principal right ideal is a maximal , then $M = aR$. Since $R$ is right SNF-ring, $R/aR$ is an N-flat right R-module .By Lemma (2.1) $a = ba$ for some $b \in aR$ .Therefore, $a = ara$ $(b = ar)$ for some $r \in R$ .By Lemma (2.7) , $a \in Ra^2 = 0$ , which implies $a = 0$ .Thus , $R$ is a reduced ring . ■

Next, we recall the following result of Wei and Chen [6] which proved the link between nil-injective and n-regular rings .

**Theorem 2.8 :**

The following conditions are equivalent for a ring $R$

1- $R$ is a n-regular ring .
2- Every left R-module is nil-injective .
3- Every cyclic left R-module is nil-injective .
4- $R$ is left nil-injective left NPP ring .

From Theorems (2.4 and 2.8) and Lemma (2.1) , we get the following theorem .

**Theorem 2.19 :**

Let $R$ be a reversible ring. Then , $R$ is a right SNF-ring ,if and only if $R$ is nil-injective .■

**3- Rings whose simple singular right R-module are N-flat**

In this section , we give an investigation of several properties for rings whose simple singular right R-modules are N-flat . Also , we study the relations between such rings and weakly regular ring .

**Definition 3.1 :**

A ring $R$ is said to be right SSNF-ring , if every simple singular right R-module is N-flat .

**Theorem 3.2 :**

If $R$ is SSNF-ring with $l(a) \subseteq r(a)$ , for every $a \in R$ then :

1- $Y(R) \cap Z(R) = 0$
2- $Y(R) \cap J(R) = 0$

**Proof :**

1) If $Y(R) \cap Z(R) \neq 0$, then there exists $0 \neq b \in Y(R) \cap Z(R)$ such that $b^2 = 0$. We claim that $RbR + r(b) = R$. Otherwise , there exists a maximal essential right ideal $M$ of $R$ containing $RbR + r(b)$. So, $R/M$ is a simple singular right R-module and

then it is right N-flat by hypothesis .Hence , $b = cb$ for some $c \in M$ (Lemma2.1) , and so $(1-b) \in l(b) \subseteq r(b) \subset M$ . Thus $1 \in M$ , which is a contradiction . Therefore $1 = x + y$, $x \in RbR$, $y \in r(b)$ and so $b = bx$. Since $RbR \subseteq Z(R)$, $x \in Z(R)$. Thus $l(1-x) = 0$ and $b = 0$, which is a contradiction . Therefore $Y(R) \cap Z(R) = 0$ .

2) Suppose $Y(R) \cap J(R) \neq 0$, there exists $0 \neq b \in J(R) \cap Y(R)$ such that $b^2 = 0$, we will prove that $RbR + r(b) = R$. If not there exists a maximal right ideal $M$ of $R$ containing $RbR + r(b)$ . Following the proof of (1) we get $b = bd$ for some $d \in RbR \subseteq J(R)$, $b(1-d) = 0$. Since $d \in J(R)$, $(1-d)$ is invertible . This implies that $b = 0$, which is a required contradiction . Therefore , $Y(R) \cap J(R) = 0$ . ■

Recall that a ring $R$ is right GQ-injective [11] if , for any right ideal $I$ isomorphic to a complement right ideal of $R$, every right R-homomorphism of $I$ into $R$ extends to an endomorphism of $R$. In [11] , shows that if $R$ is right GQ-injective ring , then $J(R) = Y(R)$, $R/J(R)$ is regular .

The next result is considered a necessary and sufficient condition for SSNF-rings to be regular ring .

**Theorem 3.3 :**

Let $R$ be reversible ring .Then , the following statements are equivalent :

1) $R$ is regular ring
2) $R$ is a right GQ-injective ring and right SSNF-ring .

**Proof :**

1 →2 Observe that if $R$ is regular then $R$ is n-regular and so every right R-module is N-flat by [9, Theorem 4.2] .So we are done .

2 →1 From Theorem (3.2) $J(R) \cap Y(R) = 0$. Since , $R$ is right GQ-injective , then $J(R) = Y(R) = 0$ and $R$ is regular ring . ■

Following [7], a ring $R$ is called strongly min-able if every right minimal idempotent element is left semicentral .

**Theorem 3.4 :**

Let $R$ be a strongly right min-able , MERT ring . If $R$ is right SSNF-ring , then $R$ is a right weakly regular ring .

**Proof :**

We shall show that $RaR + r(a) = R$, for any $a \in N(R)$. Suppose that there exists $b \in N(R)$ such that $RbR + r(b) \neq R$. Then , there exists a maximal right ideal $M$ of $R$ containing $RbR + r(b)$. If $M$ is not essential in $R$. Then , $M$ is a direct summand of $R$ because $M$ is maximal . Now , we can write $M = r(e)$ for some $0 \neq e^2 = e \in R$ and hence $eb = 0$. Because $eR$ is a minimal right ideal of $R$ and $R$ is a strongly right min-able ring , $be = ebe = 0$. Thus, $e \in r(b) \subseteq M = r(e)$, whence $e = 0$. This is a contradiction . Therefore, $M$ must be an essential right ideal of $R$. Thus , $R/M$ is N-flat and so $b = cb$ for some $c \in M$ (Lemma 2.1) , $1 \in M$ (R is MERT) . a contradiction . Therefore , $RaR + r(a) = R$. In particular $xay + z = 1$, $x, y \in R$, $z \in r(a)$. So, $axay = a$. Hence , $R$ is a weakly regular ring . ■

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