Hosoya Polynomials of Steiner Distance of the Sequential Join of Graphs

Herish O. Abdullah
herish_omer@yahoo.com
College of Sciences
University of Salahaddin

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ABSTRACT

The Hosoya polynomials of Steiner ***n***-distance of the sequential join of graphs J_3 and J_4 are obtained and the Hosoya polynomials of Steiner **3**-distance of the sequential join of ***m*** graphs J_m are also obtained.

Keywords: Steiner ***n***-distance, Hosoya polynomial, Sequential Join.

متعددات حدود هوسويا لمسافة ستينر-n لبيانات الجمع التتابعي

هيرش عبدالله

كلية العلوم، جامعة صلاح الدين

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الملخص

تضمن هذا البحث ايجاد متعددات حدود هوسويا لمسافة ستينر- ***n*** لكل من بيانات الجمع التتابعي J_3 و J_4 كما تم ايجاد متعددات حدود هوسويا لمسافة ستينر- **3** لبيان الجمع التتابعي J_m لـ ***m*** من البيانات.

الكلمات المفتاحية: مسافة ستينر-n، متعددة حدود هوسويا، الجمع التتابعي.

1. Introduction

We follow the terminology of [**2,3**]. For a connected graph $G=(V,E)$ of order ***p***, the ***Steiner distance*****[5,6,7]** of a non-empty subset $S \subseteq V(G)$, denoted by $d_G(S)$, or simply $d(S)$, is defined to be the size of the smallest connected subgraph ***T(S)*** of ***G*** that contains ***S***; ***T(S)*** is a tree called a ***Steiner tree*** of ***S***. If **|*S*|=2**, then ***d(S)*** is the distance between the two vertices of ***S***. For $2 \le n \le p$ and **|*S*|=*n***, the Steiner distance of ***S*** is called ***Steiner n-distance of S*** in ***G***. The ***Steiner n-diameter*** of ***G***, denoted by $diam_n^* G$ or simply $\delta_n^*(G)$, is defined by:

$$diam_n^* G = \max\{d_G(S) : S \subseteq V(G), |S| = n\}.$$

Remark 1.1. It is clear that

(1) If $n > m$, then $diam_n^* G \ge diam_m^* G$.

(2) If $S' \subseteq S$, then $d_G(S') \le d_G(S)$.

The Steiner n-distance of a vertex $v \in V(G)$, denoted by $W_n^*(v,G)$, is the sum of the Steiner n-distances of all n-subsets containing v. The sum of Steiner n-distances of all n-subsets of $V(G)$ is denoted by $d_n(G)$ or $W_n^*(G)$. It is clear that

$$W_n^*(G) = n^{-1} \sum_{v \in V(G)} W_n^*(v,G). \qquad ...(1.1)$$

The graph invariant $W_n^*(G)$ ***is called Wiener index of the Steiner n-distance*** of the graph G.

Definition 1.2[**1**] Let $C_n^*(G,k)$ be the number of n-subsets of distinct vertices of G with Steiner n-distance k. The graph polynomial defined by

$$H_n^*(G;x) = \sum_{k=n-1}^{\delta_n^*} C_n^*(G,k)x^k , \qquad ...(1.2)$$

where δ_n^* is the Steiner n-diameter of G; is called the ***Hosoya polynomial of Steiner n-distance of G.***

It is clear that

$$W_n^*(G) = \sum_{k=n-1}^{\delta_n^*} kC_n^*(G,k) \qquad ...(1.3)$$

For $1 \le n \le p$, let $C_n^*(u,G,k)$ be the number of n-subsets S of distinct vertices of G containing u with Steiner n-distance k. It is clear that

$C_1^*(u,G,0) = 1$.

Define

$$H_n^*(u,G;x) = \sum_{k=n-1}^{\delta_n^*} C_n^*(u,G,k)x^k . \qquad \ldots(1.4)$$

Obviously, for $2 \le n \le p$

$$H_n^*(G;x) = \frac{1}{n} \sum_{u \in V(G)} H_n^*(u,G;x). \qquad \ldots(1.5)$$

Ali and Saeed [**1**] were first who studied this distance-based graph polynomial for Steiner n-distances, and established Hosoya polynomials of Steiner n-distance for some special graphs and graphs having some kind of regularity, and for Gutman's compound graphs $G_1 \bullet G_2$ and $G_1 : G_2$ in terms of Hosoya polynomials of G_1 and G_2.

Definition 1.3[2] Let G_1, G_2, …, G_m, $m \ge 2$, be vertex disjoint graphs. The sequential join of G_1, G_2, …, G_m is a graph denoted by

$$J_m = G_1 + G_2 + ... + G_m,$$

and defined by

$$V(J_m) = \bigcup_{i=1}^{m} V(G_i),$$

$$E(J_m) = \left\{\bigcup_{i=1}^{m} E(G_i)\right\} \bigcup \left\{uv \mid u \in V_i \text{ and } v \in V_{i+1}, \text{ for } i = 1, 2, ..., m-1\right\}$$

in which $V_i = V(G_i)$, as depicted in the following figure.

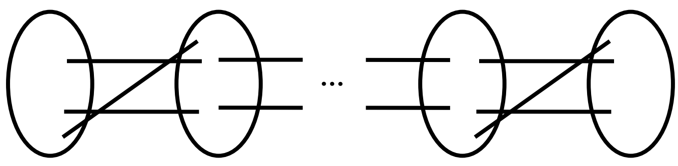

Fig. **1.1** J_m

It is clear that

$$p(J_m) = \sum_{i=1}^{m} p_i, \quad q(J_m) = q_m + \sum_{i=1}^{m-1} (q_i + p_i p_{i+1}),$$

in which

$p_i = p(G_i)$ and $q_i = q(G_i)$.

One can easily see that for $m \geq 3, \sum_{i=1}^{m} G_i$ is not commutative, that is for ***m***=**3** $G_1 + G_2 + G_3 \neq G_1 + G_3 + G_2$.

In [**8**], Saeed obtained the (ordinary) Hosoya polynomials of J_m, and in [**7**], Herish obtained the Steiner ***n***-diameter of the sequential join of ***m*** empty graphs and of ***m*** complete graphs. Also, the Hosoya polynomials of Steiner distance of the sequential join of ***m*** empty graphs and of ***m*** complete graphs were obtained. For $m \geq 3$ and $n \geq 2$, the Steiner ***n***-diameter of the sequential join of ***m*** complete graphs is given by[**7**]

$$diam_n^* J_m = \begin{cases} m + n - 3, & \text{if } 2 \leq n \leq p_1 + p_m \\ m + n - 3 - \alpha, & \text{if } p_1 + p_m + 1 \leq n \leq p, \end{cases} \quad ...(1.6)$$

where α is the smallest integer such that

$$p_1 + p_m + 1 \le n \le p_1 + p_m + \sum_{i=1}^{\alpha} r_i \ .$$

It is obvious that Eq. **1.6** holds for the sequential join of m graphs J_m.

In this paper, a generalization of the results obtained in [**7**] is given. We obtained the Hosoya polynomials of Steiner n-distance of J_3 and J_4; and the Hosoya polynomials of Steiner 3-distance of J_m, $m \ge 4$. We also obtained $H_n^*(J_3;x)$, for $n \ge 2$ and $H_3^*(J_m;x)$, for $m \ge 4$, where each of G_i, for $i = 1,2,...,m$ is a special graph.

2. Hosoya Polynomials of Steiner n-Distance of J_3 and J_4

In this section, we consider J_m, for m=**3** and m=**4**. Let S be any n-subset of vertices of J_m. Let $B(G_i)$, for $i = 1,2,...,m$, be the number of all n-subsets S such that $\langle S \rangle$ is connected in G_i. The following proposition determines the Hosoya polynomials of Steiner n-distance of J_3.

Proposition 2.1. For $3 \le n \le p(= p_1 + p_2 + p_3)$,

$$H_n^*(J_3;x) = C_1 x^{n-1} + C_2 x^n,$$

where

$$C_1 = \binom{p}{n} - \binom{p_1 + p_2}{n} - \binom{p_2}{n} + B(G_1) + B(G_2) + B(G_3),$$

$$C_2 = \binom{p_2}{n} + \binom{p_1 + p_3}{n} - \left[B(G_1) + B(G_2) + B(G_3)\right],$$

and

$B(G_1), B(G_2)$ and $B(G_3)$ are as defined above.

<u>Proof</u>. It is clear that

$$diam_n^* J_3 = \begin{cases} n, & if \quad 3 \le n \le p_1 + p_3 \\ n-1, & if \quad otherwise \end{cases} .$$

Therefore,

$$H_n^*(J_3;x) = C_1 x^{n-1} + C_2 x^n$$

in which C_1 is the number of all n-subsets of $V(J_3)$ with Steiner distance equals n-**1**, and C_2 is the number of all n-subsets of $V(J_3)$ with Steiner distance equals n.

Therefore,

$$C_2 = \sum_{i=1}^{3} \left\{ \binom{p_i}{n} - B(G_i) \right\} + \sum_{j=1}^{n-1} \binom{p_1}{j} \binom{p_3}{n-j}$$

$$=\binom{p_2}{n}+\binom{p_1+p_3}{n}-\left[B(G_1)+B(G_2)+B(G_3)\right].$$

Now, since

$$C_1+C_2=\binom{p}{n},$$

therefore

$$C_1=\binom{p}{n}-C_2=\binom{p}{n}-\binom{p_1+p_3}{n}-\binom{p_2}{n}+B(G_1)+B(G_2)+B(G_3)$$

This completes the proof. ■

The following corollary computes the n-Wiener index of J_3.

Corollary 2.2. For $3\le n\le p(=p_1+p_2+p_3)$,

$$W_n^*(J_3)=n\binom{p}{n}-C_1,$$

where C_1 is given in Proposition **2.1**. ■

Next, we shall find the Hosoya polynomials of Steiner n-distance of J_4.

Proposition 2.3. For $3\le n\le p(=p_1+p_2+p_3+p_4)$,

$$H_n^*(J_4;x)=C_1x^{n-1}+C_2x^n+C_3x^{n+1},$$

where

$$C_1=\sum_{i=1}^{n-2}\sum_{j=1}^{n-1-i}\left[\binom{p_1}{i}\binom{p_2}{j}\binom{p_3}{n-i-j}+\binom{p_2}{i}\binom{p_3}{j}\binom{p_4}{n-i-j}\right]$$
$$+\sum_{i=1}^{n-3}\sum_{j=1}^{n-2-i}\sum_{k=1}^{n-1-i-j}\binom{p_1}{i}\binom{p_2}{j}\binom{p_3}{k}\binom{p_4}{n-i-j-k}+\sum_{i=1}^{4}B(G_i)$$
$$+\binom{p_1+p_2}{n}+\binom{p_2+p_3}{n}+\binom{p_3+p_4}{n}-\binom{p_1}{n}-2\binom{p_2}{n}-2\binom{p_3}{n}-\binom{p_4}{n},$$

$$C_2=\binom{p}{n}-\sum_{i=1}^{n-2}\sum_{j=1}^{n-1-i}\left[\binom{p_1}{i}\binom{p_2}{j}\binom{p_3}{n-i-j}\right]$$
$$-\sum_{i=1}^{n-3}\sum_{j=1}^{n-2-i}\sum_{k=1}^{n-1-i-j}\binom{p_1}{i}\binom{p_2}{j}\binom{p_3}{k}\binom{p_4}{n-i-j-k}-\sum_{i=1}^{4}B(G_i)$$
$$-\binom{p_1+p_2}{n}-\binom{p_2+p_3}{n}-\binom{p_3+p_4}{n}-\binom{p_1+p_4}{n}$$

$$+2\binom{p_1}{n}+2\binom{p_2}{n}+2\binom{p_3}{n}+2\binom{p_4}{n},$$

and

$$C_3=\binom{p_1+p_4}{n}-\binom{p_1}{n}-\binom{p_4}{n}.$$

Proof. It is clear that $n-1\le diam_n^* J_4 \le n+1$, therefore the Hosoya polynomials of Steiner n-distance of J_4 has the following form

$$H_n^*(J_4;x)=C_1x^{n-1}+C_2x^n+C_3x^{n+1}.$$

To find C_1, C_2 and C_3, let S be any n-subset of vertices of J_4, then we have the following possibilities for the subset S.

(I) $d(S)=n-1$ if and only if S has any of the following subcases:

(1) S is a subset of V_i, for $i=1,2,3,4$ and $\langle S\rangle$ is a connected subgraph of G_i. The number of these n-subsets is given by

$$B(G_1)+B(G_2)+B(G_3)+B(G_4).$$

(2) $S\subseteq V_k\cup V_{k+1}$ and $(S\cap V_k\ne\varphi \wedge S\cap V_{k+1}\ne\varphi)$, $k=1,2,3$.

The number of these subsets S is given by

$$\sum_{i=1}^{n-1}\binom{p_1}{i}\binom{p_2}{n-i}+\sum_{i=1}^{n-1}\binom{p_2}{i}\binom{p_3}{n-i}+\sum_{i=1}^{n-1}\binom{p_3}{i}\binom{p_4}{n-i}$$

$$=\binom{p_1+p_2}{n}+\binom{p_2+p_3}{n}+\binom{p_3+p_4}{n}-\binom{p_1}{n}-2\binom{p_2}{n}-2\binom{p_3}{n}-\binom{p_4}{n},$$

(3) $(S\subseteq\bigcup_{i=1}^{3}V_i \wedge S\cap V_i\ne\varphi)$ or $(S\subseteq\bigcup_{i=2}^{4}V_i \wedge S\cap V_i\ne\varphi)$. The number of these n-subsets is given by

$$\sum_{i=1}^{n-2}\sum_{j=1}^{n-1-i}\left[\binom{p_1}{i}\binom{p_2}{j}\binom{p_3}{n-i-j}+\binom{p_2}{i}\binom{p_3}{j}\binom{p_4}{n-i-j}\right]$$

(4) $S\cap V_i\ne\varphi$, $i=1,2,3,4$. The number of these n-subsets is given by

$$\sum_{i=1}^{n-3}\sum_{j=1}^{n-2-i}\sum_{k=1}^{n-1-i-j}\binom{p_1}{i}\binom{p_2}{j}\binom{p_3}{k}\binom{p_4}{n-i-j-k}$$

From (**1**), (**2**), (**3**) and (**4**), we get C_1 as given in the statement of the proposition.

(II) $d(S)=n+1$ if and only if $S\subseteq V_1\cup V_4$ and ($S\cap V_1\ne\varphi$ and $S\cap V_4\ne\varphi$). The number of these S's is given by

$$\sum_{i=1}^{n-1}\binom{p_1}{i}\binom{p_4}{n-i}=\binom{p_1+p_4}{n}-\binom{p_1}{n}-\binom{p_4}{n}.$$

So, C_3 is as given.

Now, since $C_1+C_2+C_3=\binom{p}{n}$,

therefore

$$C_2=\binom{p}{n}-C_1-C_3.$$

This completes the proof. ■

Remark. The triple summation in C_1 is taken to be zero when ***n*****=3**.

The following corollary computes $W_n^*(J_4)$.

Corollary 2.4. For $3\le n\le p(=p_1+p_2+p_3+p_4)$,

$$W_n^*(J_4)=n\binom{p}{n}-C_1+C_3,$$

where C_1 and C_3 are given in Proposition **2.3.**■

Remark. For $m\ge 5$, the calculation of the coefficients of $H_n^*(J_m;x)$ is complicated.

The numbers $B(G_1)$, $B(G_2)$ and $B(G_3)$ are given in Proposition **2.1** can be counted for some specific graphs G_1, G_2 and G_3 as in the following examples.

Example 2.5. Let N_{p_1}, N_{p_2} and N_{p_3} be empty graphs of orders p_1, p_2 and p_3 respectively, then

$$B(N_{p_1})=B(N_{p_2})=B(N_{p_3})=0.$$

Example 2.6. Let K_{p_1}, K_{p_2} and K_{p_3} be complete graphs of orders p_1, p_2 and p_3 respectively, then

$$B(K_{p_i})=\binom{p_i}{n},\ \text{for } i=1,2,3.$$

Example 2.7. Let P_{α_1}, P_{α_2} and P_{α_3} be path graphs of orders α_1, α_2 and α_3 respectively, then

$$B(P_{\alpha_i})=\alpha_i-n+1,\ \text{for } i=1,2,3.$$

Example 2.8. Let C_{α_1}, C_{α_2} and C_{α_3} be cycle graphs of orders α_1, α_2 and α_3 respectively, then

$$B(C_{\alpha_i}) = \alpha_i, \text{ for } i = 1,2,3.$$

Example 2.9. Let W_{α_1}, W_{α_2} and W_{α_3} be wheel graphs of orders α_1, α_2 and α_3 respectively, then

$$B(W_{\alpha_i}) = \binom{\alpha_i - 1}{n-1} + \alpha_i - 1, \text{ for } i = 1,2,3.$$

Example 2.10. Let K_{α_i,β_i}, for $i = 1,2,3$, be complete bipartite graphs of partite sets of size α_i β_i , then

$$B(K_{\alpha_i,\beta_i}) = \binom{\alpha_i + \beta_i}{n} - \binom{\alpha_i}{n} - \binom{\beta_i}{n}, \text{ for } i = 1,2,3.$$

3. Hosoya Polynomials of Steiner 3-Distance of J_m $(m \geq 5)$

In this section, we consider $J_m = G_1 + G_2 + ... + G_m$, for $m \geq 5$. The following theorem determines Hosoya polynomials of Steiner **3**-distance of J_m.

Theorem 3.1. For $m \geq 5$,

$$H_3^*(J_m;x) = (A + Bx)x^2 + \tfrac{1}{2}\sum_{j=i+1}^{m}\sum_{i=1}^{m-1} p_i p_j (p_i + p_j - 2)x^{j-i+1}$$
$$+ \sum_{j=i+2}^{m}\sum_{i=1}^{m-2} p_i p_j \left(\sum_{r=1}^{j-1} p_r\right) x^{j-i},$$

where

$$A = \sum_{i=1}^{m}\left[\sum_{v\in V_i}\binom{\deg v}{2} - 2\gamma_i\right], \quad B = \sum_{i=1}^{m}\binom{p_i}{3} - A,$$

in which γ_i, for $i = 1,2,...,m$ is the number of non-identical triangles K_3 as a subgraph in G_i.

<u>**Proof.**</u> Let S be any **3**-subset of vertices of J_m, then we have three main cases for the subset S.

(I) If $S \subseteq V_i$, for $i = 1,2,...,m$, then

(a) $d(S) = 2$, when $\langle S \rangle$ is a connected subgraph in G_i, and by Lemma **3.4.4.** of [7], the number of such **3**-subsets S is given by

$$A = \sum_{i=1}^{m} \left[\sum_{v \in V_i} \binom{\deg v}{2} - 2\gamma_i \right].$$

(b) $d(S) = 3$, when $\langle S \rangle$ is a disconnected subgraph in G_i, and the number of such **3**-subsets ***S*** is given by

$$B = \sum_{i=1}^{m} \binom{p_i}{3} - A.$$

Case(**I**) produces the polynomial

$$F_1(x) = (A + Bx)x^2.$$

(II) Either two vertices of S are in V_i and one vertex of ***S*** in V_j, $i < j$, or one vertex of ***S*** in V_i, and two vertices of ***S*** in V_j, for $1 \le i < j \le m$. For each such cases of ***S***,

$$d(S) = j - i + 1,$$

and the number of ways of choosing such ***S*** is given by

$$\sum_{j=i+1}^{m} \sum_{i=1}^{m-1} \left[\binom{p_i}{2} p_j + \binom{p_j}{2} p_i \right],$$

and, this produces the polynomial

$$F_2(x) = \frac{1}{2} \sum_{j=i+1}^{m} \sum_{i=1}^{m-1} [p_j p_i (p_i - 1) + p_i p_j (p_j - 1)] x^{j-i+1}$$

$$= \frac{1}{2} \sum_{j=i+1}^{m} \sum_{i=1}^{m-1} p_i p_j (p_i + p_j - 2) x^{j-i+1}$$

(III) One vertex of S in V_i, one vertex in V_j, $j \ge i + 2$, and the third vertex in V_r, $i < r < j$. For such case

$$d(S) = j - i,$$

and the number of all possibilities of such ***S*** is

$$\sum_{j=i+2}^{m} \sum_{i=1}^{m-2} p_i p_j \left(\sum_{r=i+1}^{j-1} p_r \right),$$

and this produces the polynomial

$$F_3(x) = \sum_{j=i+2}^{m} \sum_{i=1}^{m-2} p_i p_j \left(\sum_{r=i+1}^{j-1} p_r \right) x^{j-i}.$$

Now adding the polynomials $F_1(x), F_2(x)$ and $F_3(x)$ obtained in **(I)**, **(II)** and **(III)**, we get the required result. ■

The numbers ***A*** and ***B*** are given in Theorem **3.1** can be counted when G_i, for $i = 1, 2, ..., m$, has a special form, as in the following examples.

Example 3.2. Let N_{p_i}, for $i=1,2,...,m$ be empty graphs of orders p_i, then

$$A=0 \text{ and } B=\sum_{i=1}^{m}\binom{p_i}{3}.$$

Example 3.3. Let K_{p_i}, for $i=1,2,...,m$ be complete graphs of orders p_i, then

$$A=\sum_{i=1}^{m}\binom{p_i}{3} \text{ and } B=0.$$

Example 3.4. Let P_{α_i}, for $i=1,2,...,m$ be path graphs of orders α_i, then

$$A=\sum_{i=1}^{m}[\alpha_i-2]=p-2m \text{ and } B=\sum_{i=1}^{m}\binom{\alpha_i}{3}-p+2m\cdot$$

Example 3.5. Let C_{α_i}, for $i=1,2,...,m$ be cycle graphs of orders α_i, then

$$A=\sum_{i=1}^{m}\alpha_i=p \text{ and } B=\sum_{i=1}^{m}\binom{\alpha_i}{3}-p\cdot$$

Example 3.6. Let W_{α_i} for $i=1,2,...,m$ be wheel graphs of orders α_i, then

$$A=\sum_{i=1}^{m}\left[\sum_{v\in V_i}\binom{\deg v}{2}-2\gamma_i\right]=\sum_{i=1}^{m}\left[(\alpha_i-1)\binom{3}{2}+\binom{\alpha_i-1}{2}-2(\alpha_i-1)\right]$$

$$=\sum_{i=1}^{m}\binom{\alpha_i}{2},$$

and

$$B=\sum_{i=1}^{m}\binom{\alpha_i}{3}-\sum_{i=1}^{m}\binom{\alpha_i}{2}=\tfrac{1}{6}\sum_{i=1}^{m}\alpha_i(\alpha_i-1)(\alpha_i-5)\cdot$$

Example 3.7. Let K_{α_i,β_i}, for $i=1,2,...,m$, be complete bipartite graphs of partite sets of size α_i β_i, then it is known that K_{α_i,β_i} contains no odd cycles, and so $\gamma_i=0$, for $i=1,2,...,m$.

Hence,

$$A=\sum_{i=1}^{m}\left[\alpha_i\binom{\beta_i}{2}+\beta_i\binom{\alpha_i}{2}\right]=\tfrac{1}{2}\sum_{i=1}^{m}\alpha_i\beta_i(\alpha_i+\beta_i-2),$$

and

$$B=\sum_{i=1}^{m}\left[\binom{\alpha_i+\beta_i}{3}-\tfrac{1}{2}\alpha_i\beta_i(\alpha_i+\beta_i-2)\right]\cdot$$

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