Numerical Solution and Stability Analysis for Burger's-Huxley Equation

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Received on: 02/09/2007 **Accepted on: 17/12/2007**

ABSTRACT

The Burger's-Huxley equation has been solved numerically by using two finite difference methods, the explicit scheme and the Crank-Nicholson scheme. A comparison between the two schemes has been made and it has been found that, the first scheme is simpler while the second scheme is more accurate and has faster convergent. Also, the stability analysis of the two methods by using Fourier (Von Neumann) method has been done and the results were that, the explicit scheme is stable under the condition $\Delta t \le \frac{2(\Delta x)^2}{4\nu + \delta\beta(\Delta x)^2}$ and the Crank-Nicholson is unconditionally stable.

Keywords: Stability Analysis, Explicit scheme, Crank-Nicholson scheme, Burger's-Huxley Equation, Fourier (Von Neumann) method.

الحل العددي وتحليل الاستقرار لمعادلة Burger's-Huxley

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تاريخ الاستلام: 2007/09/02 **تاريخ القبول: 2007/12/17**

الملخص

لقد تم حل معادلة Burger's-Huxley عدديا باستخدام طريقتين من طرائق الفروقات المنتهية، الأولى هي الطريقة الصريحة (Explicit scheme) والثانية هي طريقة (Crank-Nicholson) اذ تم عمل مقارنة بين نتائج كلتا الطريقتين، وقد تبين أن الطريقة الأولى هي الأسهل في حين كانت الطريقة الثانية أدق وأسرع تقارباً. لقد تمت كذلك دراسة الاستقرارية العددية للطريقتين المستخدمتين في حل معادلة Burger's-Huxley باستخدام طريقة (Von-Neumann) Fourier، اذ تبين ان الطريقة الأولى مستقرة تحت الشرط $\Delta t \le \frac{2(\Delta x)^2}{4\nu + \delta\beta(\Delta x)^2}$ بينما الطريقة الثانية مستقرة من دون الشروط .

الكلمات المفتاحية: تحليل الاستقراية، الطريقة الصريحة، طريقة (Crank-Nicholson)، معادلة Burger's-Huxley، طريقة (Von-Neumann) Fourier.

1. Introduction

Nowadays engineers and scientists in all fields of their research are using partial differential equations to describe their problems and thus such partial differential equations arise in the study of heat transfer, boundary-layer flow, fluid flow problems, vibrations elasticity, circular and rectangular wave guides, in applied mathematics and so on.

Finding the exact solution for the above problems which involve partial differential equations is difficult in some cases. Hence we have to find the numerical solution of these problems using computers which came into existence[7].

Parabolic PDEs describe practically useful phenomena such as transport chemistry problems of the advection-diffusion-reaction type and problem of this type plays an important role in the modeling of pollution of the atmosphere, ground water and surface water [3].

For time-dependent problems considerable progress in Finite difference methods was made during the period of, and immediately following, the Second World War, when large-scale practical applications became possible with the aid of computers. A major role was played by the work of Von Neumann, partly reported in O'Brien, Hyman and Kaplan (1951). For parabolic equations a highlight of the early theory was the important paper by John (1952). For mixed initial–boundary value problems the use of implicit methods was also established in this period by Crank and Nicholson (1947).[9].

2. Mathematical model

We consider The generalized Burger's-Huxley equation [10] of the form

$$\frac{\partial u}{\partial t}+\alpha u^s\frac{\partial u}{\partial x}-\nu\frac{\partial^2 u}{\partial x^2}=\beta u(1-u^s)(u^s-\delta),\qquad 0\le x\le 1,\ t\ge 0 \qquad \ldots(1)$$

$$\frac{\partial u}{\partial t}+\alpha u^s\frac{\partial u}{\partial x}=\beta u(1-u^s)(u^s-\delta),\qquad 0\le x\le 1,\ t\ge 0 \qquad \ldots(2)$$

The equation (1) represents the generalized Burger's-Huxley equation with diffusion term and the equation (2) represents the generalized Burger's-Huxley equation without diffusion term.

With the initial condition

$$u(x,0)=\left(\frac{\delta}{2}+\frac{\delta}{2}\tanh[A_1x]\right)^{1/s}=f(x) \qquad \ldots(3)$$

and boundary conditions

$$u(0,t)=\left(\frac{\delta}{2}+\frac{\delta}{2}\tanh[-A_1A_2t]\right)^{1/s}=g_0(t),\ t\ge 0 \qquad \ldots(4)$$

and

$$u(1,t)=\left(\frac{\delta}{2}+\frac{\delta}{2}\tanh[A_1(1-A_2t)]\right)^{1/s}=g_1(t),\ t\geq 0 \qquad \ldots(5)$$

The Solitary wave solution of Eq. (1) is

$$u(x,t)=\left(\frac{\delta}{2}+\frac{\delta}{2}\tanh[A_1(x-A_2t)]\right)^{1/s}$$

where

$$A_1=\frac{-\alpha s+s\sqrt{\alpha^2+4\beta(1+s)}}{4(1+s)}\delta,$$

$$A_2=\frac{\delta\alpha}{1+s}-\frac{(1+s-\delta)(-\alpha+\sqrt{\alpha^2+4\beta(1+s)})}{2(1+s)}$$

where v is diffusion coefficient α,β,δ and s are parameters that $\beta\geq 0$, $s>0$, $\delta\in(0,1)$.[2]

2.1 Finite Difference Approximations

Let the interval [a,b] be divided into N equal subintervals with the length of each subinterval, called the ***grid size***, given as

$$\Delta x=\frac{b-a}{N}$$

We define the point p as the point having the coordinate $p(\Delta x)$, and denote that point x_p; that is, $x_p=p(\Delta x)$ *for* $p=0,1,2,.....,N$

This means that $x_0=a,\quad x_N=b,\quad x_{p+1}=(p+1)\Delta x,\quad x_{p-1}=(p-1)\Delta x,\quad etc.$

The variable y corresponding to the point x_p is denoted as y_p that is, $y(x_p)=y_p$.The three formulas for approximating first derivatives are:

$$(\frac{dy}{dx})_p\approx\frac{y_{p+1}-y_p}{\Delta x} \qquad \ldots(6)$$

$$(\frac{dy}{dx})_p\approx\frac{y_p-y_{p-1}}{\Delta x} \qquad \ldots(7)$$

$$(\frac{dy}{dx})_p\approx\frac{y_{p+1}-y_{p-1}}{2\Delta x} \qquad \ldots(8)$$

The three previous equations are called, ***forward difference approximation, backward difference approximation*** and ***central difference approximation***, respectively. And the approximating formula for the second derivative is[6]:

$$(\frac{d^2y}{dx^2})_p\approx\frac{y_{p+1}-2y_p+y_{p-1}}{(\Delta x)^2} \qquad \ldots(9)$$

2.2 Grid points:

Let u be a function of independent variables x and t . The (x,t)-plane be divided into a network of rectangles of sides $\Delta x=h$ and $\Delta t=k$ by drawing the set of lines .

$$\left.\begin{array}{l} x = ph, \quad p = 0,1,2,3,...... \\ t = qk, \quad q = 0,1,2,3,...... \end{array}\right\} \qquad ...(10)$$

The point of intersection of these families of lines are called '**Grid Point**' (or some times referred to as **Lattice Points** , **Mesh points**) . [7]

The general procedure to solve the partial differential equations by finite-difference approximation is to obtain the solution at these grid points. In view of the lines $x = ph$, $t = qk$ defined above, we can rewrite the finite difference approximations to the first and second derivatives as follows:

The first order derivative of u w.r.t. x is given by:

$$u_x(x_p, t_q) = \frac{u_{p+1,q} - u_{p,q}}{h} + O(h) \qquad ...(11)$$

$$u_x(x_p, t_q) = \frac{u_{p,q} - u_{p-1,q}}{h} + O(h) \qquad ...(12)$$

$$u_x(x_p, t_q) = \frac{u_{p+1,q} - u_{p-1,q}}{2h} + O(h^2) \qquad ...(13)$$

and $$u_{xx}(x_p, t_q) = \frac{u_{p+1,q} - 2u_{p,q} + u_{p-1,q}}{h^2} + O(h^2) \qquad ...(14)$$

Similarly, w.r.t. the independent variable t , we have

$$u_t(x_p, t_q) = \frac{u_{p,q+1} - u_{p,q}}{k} + O(k) \qquad ...(15)$$

$$u_t(x_p, t_q) = \frac{u_{p,q} - u_{p,q-1}}{k} + O(k) \qquad ...(16)$$

$$u_t(x_p, t_q) = \frac{u_{p,q+1} - u_{p,q-1}}{2k} + O(k^2) \qquad ...(17)$$

and $$u_{tt}(x_p, t_q) = \frac{u_{p,q+1} - 2u_{p,q+} u_{p,q-1}}{k^2} + O(k^2) \qquad ...(18)$$

In the (x,t)-plane the above derivatives can be analyzed as follows: [7]

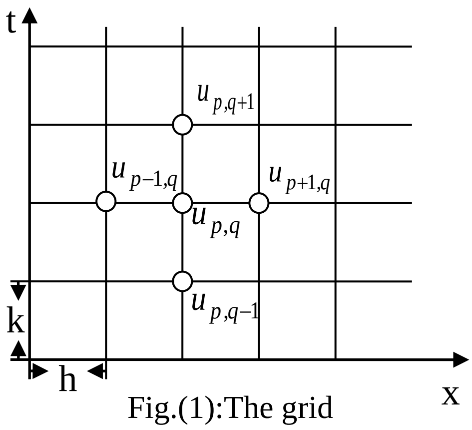

Fig.(1):The grid

2.3 Derivation of Explicit Method For Burger's-Huxley Equation

Assume that the rectangle $R=\{(x,t):0\le x\le a,\quad 0\le t\le b\}$ is subdivided into n-1 by m-1 rectangle with sides $\Delta x=h$ and $\Delta t=k$, as shown in Figure(1). Start at the bottom row, where $t=t_1=0$, and the solution is $u(x_p,t_1)=f(x_p)$.

A method for computing the approximations to $u(x,t)$ at grid points in successive rows $\{u(x_p,t_q):p=1,2,...,n\}$, for $q=2,3,...,m$.

The grid spacing is uniform in every row: $x_{p+1}=x_{p+h}$ and ($x_{p-1}=x_{p-h}$), and it is uniform in every column: $t_{q+1}=t_{q+k}$ and ($t_{q-1}=t_{q-k}$) .Next, we drop the terms $O(k),O(h) and O(h^2)$ [5], and use the approximation $u_{p,q}$ for $u(x_p,t_q)$ in equation (11),(15) and substituted into equation (2) when $\delta=1$,to obtain

$$\frac{u_{p,q+1}-u_{p,q}}{k}+\alpha u_{p,q}\frac{u_{p+1,q}-u_{p,q}}{h}=\beta u_{p,q}(1-u_{p,q})(u_{p,q}-\delta)$$

$$u_{p,q+1}-u_{p,q}+\frac{\alpha k}{h}u_{p,q}.u_{p+1,q}-\frac{\alpha k}{h}u^2{}_{p,q}=k\beta u_{p,q}(1-u_{p,q})(u_{p,q}-\delta)$$

Let $r=\frac{\alpha k}{h}$

$$\Rightarrow u_{p,q+1}=u_{p,q}-ru_{p,q}.u_{p+1,q}+ru^2{}_{p,q}+k\beta u^2{}_{p,q}-k\beta\delta u_{p,q}-k\beta u^3{}_{p,q}+k\beta\delta u^2{}_{p,q}$$

$$\Rightarrow u_{p,q+1}=(1-k\beta\delta-ru_{p+1,q})u_{p,q}+(r+k\beta+k\beta\delta)u^2{}_{p,q}-k\beta u^3{}_{p,q} \qquad \ldots(19)$$

The result is the explicit forward difference equation to the Burger's-Huxley equation without diffusion term.

Also, we drop the terms $O(k),O(h) and O(h^2)$, and use the approximation $u_{p,q}$ for $u(x_p,t_q)$ in equation (11),(14) and(15) and substituted into equation (1) to obtain

$$\frac{u_{p,q+1}-u_{p,q}}{k}+\alpha u_{p,q}\frac{u_{p+1,q}-u_{p,q}}{h}-\nu\frac{u_{p-1,q}-2u_{p,q}+u_{p+1,q}}{h^2}=\beta u_{p,q}(1-u_{p,q})(u_{p,q}-\delta)$$

$$u_{p,q+1}-u_{p,q}+\frac{\alpha k}{h}u_{p,q}u_{p+1,q}-\frac{\alpha k}{h}u^2{}_{p,q}-\frac{\nu k}{h^2}(u_{p-1,q}-2u_{p,q}+u_{p+1,q})=k\beta u_{p,q}(1-u_{p,q})(u_{p,q}-\delta)$$

let $r=\frac{k}{h}$ and , $m_1=\frac{\nu k}{h^2}$ we get

$$\begin{aligned}u_{p,q+1}=&m_1u_{p-1,q}+(1-2m_1-k\beta\delta)u_{p,q}+(\alpha r+k\beta+k\beta\delta)u^2_{p,q}-k\beta u^3_{p,q}\\&+(m_1-\alpha ru_{p,q})u_{p+1,q}\end{aligned} \qquad ...(20)$$

The result is the explicit forward difference equation to the Burger's-Huxley equation with diffusion term.

2.4 Derivation of Crank-Nicholson Method for Burger's-Huxley Equation

The second order derivative (diffusion term) in Eq (1) is replaced by the average of its finite-difference of second order approximation at q^{th} and $(q+1)^{th}$ times rows. This method was invented by John Crank and Phyllis Nicholson in (1947), [7] then the equation (1) is approximated as

$$\frac{u_{p,q+1}-u_{p,q}}{k}+\alpha u_{p,q}\frac{u_{p+1,q}-u_{p,q}}{h}-\nu[\frac{u_{p-1,q}-2u_{p,q}+u_{p+1,q}}{2h^2}$$
$$+\frac{u_{p-1,q+1}-2u_{p,q+1}+u_{p+1,q+1}}{2h^2}]=\beta u_{p,q}(1-u_{p,q})(u_{p,q}-\delta)$$

$$u_{p,q+1}-u_{p,q}+\frac{\alpha k}{h}u_{p,q}.u_{p+1,q}-\frac{\alpha k}{h}u^2{}_{p,q}-\frac{\nu k}{2h^2}[u_{p-1,q}-2u_{p,q}+u_{p+1,q}$$
$$+u_{p-1,q+1}-2u_{p,q+1}+u_{p+1,q+1}]=(k\beta u_{p,q}-k\beta u^2{}_{p,q})(u_{p,q}-\delta)$$

Let $r=\frac{k}{h}$ and $m_2=\frac{\nu r}{2h}$

$$\Rightarrow u_{p,q+1}-u_{p,q}+\alpha ru_{p,q}.u_{p+1,q}-\alpha ru^2{}_{p,q}-m_2u_{p-1,q}+2m_2u_{p,q}-m_2u_{p+1,q}$$
$$-m_2u_{p-1,q+1}+2m_2u_{p,q+1}-m_2u_{p+1,q+1}=k\beta u^2{}_{p,q}-k\beta\delta u_{p,q}-k\beta u^3{}_{p,q}+k\beta\delta u^2{}_{p,q}$$
$$\Rightarrow -m_2u_{p-1,q+1}+(1+2m_2)u_{p,q+1}-m_2u_{p+1,q+1}=m_2u_{p-1,q}+(1-2m_2-k\beta\delta)u_{p,q}$$
$$+(\alpha r+k\beta+k\beta\delta)u^2_{p,q}-k\beta u^3_{p,q}+(m_2-\alpha ru_{p,q})u_{p+1,q} \quad \ldots(21)$$

The result is the crank-Nicholson equation to the Burger's-Huxley equation with diffusion term.

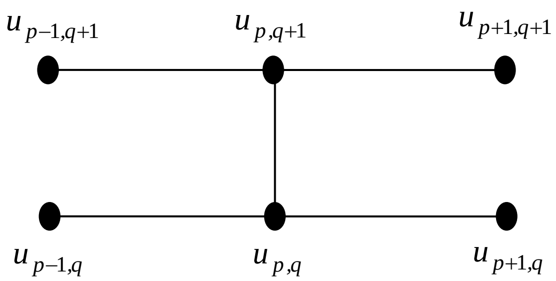

Fig.(2):The crank-Nicholson stencil.

In this time we must solve for the three values $u_{p-1,q+1}, u_{p,q+1}$ and $u_{p+1,q+1}$ For p=2,3,...,n-1.The terms on the right-hand side of equation (21) are all known. Hence the equations in (21) form a tridiagonal linear system

AX=B …(22)

The boundary conditions are used in the first and last equation(i.e. $u_{1,q} = u_{1,q+1} = g_0(t)$ and $u_{n,q} = u_{n,q+1} = g_1(t)$, respectively). The equations (21) are especially pleasing to view in their tridiagonal matrix form ***AX=B*** .

$$\begin{bmatrix} (1+2m_2) & -m_2 & 0 & 0 & 0 & 0 & 0 \\ -m_2 & (1+2m_2) & -m_2 & 0 & 0 & 0 & 0 \\ 0 & -m_2 & (1+2m_2) & -m_2 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -m_2 & (1+2m_2) & -m_2 & 0 \\ 0 & 0 & 0 & 0 & -m_2 & (1+2m_2) & -m_2 \\ 0 & 0 & 0 & 0 & 0 & -m_2 & (1+2m_2) \end{bmatrix} . \begin{bmatrix} u_{2,q+1} \\ u_{3,q+1} \\ u_{4,q+1} \\ \vdots \\ u_{n-3,q+1} \\ u_{n-2,q+1} \\ u_{n-1,q+1} \end{bmatrix} =$$

$$\begin{bmatrix} m_2u_{1,q+1} + m_2u_{1,q} + (1-2m_2 - k\beta\delta)u_{2,q} + (\alpha r + k\beta + k\beta\delta)u^2_{2,q} - k\beta u^3_{2,q} + (m_2 - \alpha r u_{2,q})u_{3,q} \\ m_2u_{2,q} + (1-2m_2 - k\beta\delta)u_{3,q} + (\alpha r + k\beta + k\beta\delta)u^2_{3,q} - k\beta u^3_{3,q} + (m_2 - \alpha r u_{3,q})u_{4,q} \\ m_2u_{3,q} + (1-2m_2 - k\beta\delta)u_{4,q} + (\alpha r + k\beta + k\beta\delta)u^2_{4,q} - k\beta u^3_{4,q} + (m_2 - \alpha r u_{4,q})u_{5,q} \\ \vdots \\ \vdots \\ m_2u_{n-3,q} + (1-2m_2 - k\beta\delta)u_{n-2,q} + (\alpha r + k\beta + k\beta\delta)u^2_{n-2,q} - k\beta u^3_{n-2,q} + (m_2 - \alpha r u_{n-2,q})u_{n-1,q} \\ m_2u_{n-2,q} + (1-2m_2 - k\beta\delta)u_{n-1,q} + (\alpha r + k\beta + k\beta\delta)u^2_{n-1,q} - k\beta u^3_{n-1,q} + (m_2 - \alpha r u_{n-1,q})u_{n,q} + m_2u_{n,q+1} \end{bmatrix}$$

When the Crank-Nicholson method is implemented with a computer, the linear system ***AX=B*** can be solved by either direct methods or by iterations method.

We use the direct methods (Gaussian Elimination Method) to solve the linear system in eq.(22) in this study.

3. Stability Analysis

The errors are introduced by the truncation of the series which are used to represent the derivatives in the process of replacing the differential equations by finite difference equation. We are interested in studying the growth of these errors and find the conditions for which the errors will be minimized from one time step to the next. The solution obtained by numerical methods definitely differs from the exact solution but this quantum of difference is most important.

The difference between the numerical solution and the exact solution, at any given step is known as the total error at a particular step. The most important aspect of the numerical methods, with so many different techniques available to solve a differential equation, is to minimize this error and simultaneously obtain the solutions with least error. The total error at any stage is the sum of round-off error and truncation error. [7]

3.1 Stability by the Fourier series method (Von Neumann's method)

This method, developed by Von Neumann during World War II, was first discussed in detail by O'Brien, Hyman and Kaplan in a paper published in 1951,[8].An expression of the Fourier series as $\sum a_n \cos(cx)$ or $\sum a_n \sin(cx)$, where c is some constant. Alternatively, this expressed in complex exponential form is taken in the form, $e^{i\gamma x}$ where γ is a positive constant and $i=\sqrt{-1}$. We then assure $\psi(t)e^{i\gamma x}$ as a solution of the difference equation.

The general principle for this method is replacement of the solution given by finite difference method at time t by the value $\psi(t)e^{i\gamma x}$ [7]

3.2 Stability Analysis of Explicit Method by Fourier (Von Neumann) Method

To apply Von Neumann's method on equation (2), we go to linearized stability analysis [1] and get after we eliminate the non linear term

$$\frac{\partial u}{\partial t} = -\delta\beta u \qquad ...(23)$$

By using the explicit method for equation (23) we get

$$\frac{u_{p,q+1} - u_{p,q}}{\Delta t} = -\delta\beta u_{p,q} \qquad ...(24)$$

Substituting $u_{p,q}$ by $\psi(t)e^{i\gamma x}$ in equation (24) yields

$$\frac{\psi(t+\Delta t)e^{i\gamma x} - \psi(t)e^{i\gamma x}}{\Delta t} = -\delta\beta\psi(t)e^{i\gamma x}$$

$\left[\frac{\psi(t+\Delta t)-\psi(t)}{\Delta t}\right]e^{i\gamma x} = -\delta\beta\psi(t)e^{i\gamma x}$, dividing both sides by $e^{i\gamma x}$, we get

$$\psi(t+\Delta t)-\psi(t) = -\delta\beta\Delta t\psi(t)$$

$$\psi(t+\Delta t) = [1-\delta\beta\Delta t]\psi(t)$$

$$\Rightarrow \frac{\psi(t+\Delta t)}{\psi(t)} = [1-\delta\beta\Delta t] = \xi \qquad ...(25)$$

Where ξ is An amplification factor, the solution from a particular plane $\psi(t)$ to the next plane $\psi(t+\Delta t)$, $|\psi(t+\Delta t)-\psi(t)|$ must start decreasing or alternatively $\psi(t)$ must be a bounded function, *i.e.* $\psi(t)$ should not tend to infinity for large t .From eq.(25), for boundedness of $\psi(t)$, we need

$$|\psi(t+\Delta t)/\psi(t)| \le 1$$

i.e. $$|\xi| \le 1$$

$$|1-\delta\beta\Delta t| \le 1 \qquad ...(26)$$

$$-1 \le 1 - \delta\beta\Delta t \le 1$$

In above inequality, the right-side inequality is

$$1 - \delta\beta\Delta t \le 1$$

This is always true. Hence, in order that Eq (26) is to be satisfied, we need

$$-1 \le 1 - \delta\beta\Delta t$$

$$\Rightarrow \Delta t \le \frac{2}{\delta\beta} \qquad ...(27)$$

The inequality (27) represents the imposed condition for explicit method for Burger's- Huxley equation without diffusion term to be stable.

Now, the stability analysis of explicit method by Fourier (Von Neumann) on eq.(1), also we go to linearized stability analysis and get after we eliminate the non linear term to obtain

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} = -\delta\beta u \qquad ...(28)$$

By using the explicit method for equation (28) we get

$$\frac{u_{p,q+1} - u_{p,q}}{\Delta t} - \nu \frac{u_{p-1,q} - 2u_{p,q} + u_{p+1,q}}{(\Delta x)^2} = -\delta\beta u_{p,q} \qquad ...(29)$$

Substituting $u_{p,q}$ by $\psi(t)e^{i\gamma x}$ in equation (29) yields

$$\frac{\psi(t+\Delta t)e^{i\gamma x} - \psi(t)e^{i\gamma x}}{\Delta t} - \nu \frac{\psi(t)e^{i\gamma(x-\Delta x)} - 2\psi(t)e^{i\gamma x} + \psi(t)e^{i\gamma(x+\Delta x)}}{(\Delta x)^2} = -\delta\beta\psi(t)e^{i\gamma x}$$

$$\left[\frac{\psi(t+\Delta t) - \psi(t)}{\Delta t}\right]e^{i\gamma x} - \frac{\nu\psi(t)}{(\Delta x)^2}\left[e^{-i\gamma\Delta x} - 2 + e^{i\gamma\Delta x)}\right]e^{i\gamma x} = -\delta\beta\psi(t)e^{i\gamma x} \qquad ...(30)$$

Dividing both sides in above equation by $e^{i\gamma x}$ to obtain

$$\psi(t+\Delta t) - \psi(t) - \frac{\nu\Delta t\psi(t)}{(\Delta x)^2}\left[e^{-i\gamma\Delta x} - 2 + e^{i\gamma\Delta x)}\right] = -\delta\beta\Delta t\psi(t)$$

Assume that $r = \frac{\nu\Delta t}{(\Delta x)^2}$

$$\Rightarrow \psi(t+\Delta t) - \psi(t) = r\psi(t)\left[e^{-i\gamma\Delta x} + e^{i\gamma\Delta x)} - 2\right] - \delta\beta\Delta t\psi(t)$$

$$\psi(t+\Delta t) - \psi(t) = r\psi(t)\left[\cos\gamma\Delta x - i\sin\gamma\Delta x + \cos\gamma\Delta x + i\sin\gamma\Delta x - 2\right] - \delta\beta\Delta t\psi(t)$$

$$\psi(t+\Delta t) - \psi(t) = 2r\psi(t)\left[\cos\gamma\Delta x - 1\right] - \delta\beta\Delta t\psi(t)$$

$$\Rightarrow \psi(t+\Delta t) = \psi(t) - 4r\psi(t)\sin^2(\gamma\Delta x/2) - \delta\beta\Delta t\psi(t)$$

$$\frac{\psi(t+\Delta t)}{\psi(t)} = [1 - 4r\sin^2(\gamma\Delta x/2) - \delta\beta\Delta t] = \xi \qquad ...(31)$$

Where ξ is an amplification factor, the necessary and sufficiently condition for numerical stability is $|\xi| \le 1$ that is

$$\left|\frac{\psi(t+\Delta t)}{\psi(t)}\right| \le 1$$

$$\Rightarrow \left|1-4r\,sin^2(\gamma\Delta x/2)-\delta\beta\Delta t\right| \le 1 \qquad ...(32)$$

$$\Rightarrow -1 \le 1-4r\,sin^2(\gamma\Delta x/2)-\delta\beta\Delta t \le 1$$

In above inequality, the right-side inequality is

$$1-4r\,sin^2(\gamma\Delta x/2)-\delta\beta\Delta t \le 1$$

$$\Rightarrow -4r\,sin^2(\gamma\Delta x/2) \le \delta\beta\Delta t$$

Since $r=\dfrac{\nu\Delta t}{(\Delta x)^2} \ge 0$ and this is always true.

Hence, in order that Eq(32) is to be satisfied, we need

$$-1 \le 1-4r\,sin^2(\gamma\Delta x/2)-\delta\beta\Delta t$$

$$2 \ge 4r\,sin^2(\gamma\Delta x/2)+\delta\beta\Delta t$$

$$\frac{1}{2}-\frac{\delta\beta\Delta t}{4} \ge r\,sin^2(\gamma\Delta x/2)$$

For some β, $sin^2(\gamma\Delta x/2)$ is unity and hence the above condition reduces to

$$r \le \frac{1}{2}-\frac{\delta\beta\Delta t}{4}$$

$$r \le \frac{2-\delta\beta\Delta t}{4} \qquad ...(33)$$

Since $r=\dfrac{\nu\Delta t}{(\Delta x)^2}$, implies $\dfrac{\nu\Delta t}{(\Delta x)^2} \le \dfrac{2-\delta\beta\Delta t}{4}$

$$\Rightarrow \Delta t \le \frac{2(\Delta x)^2}{4\nu+\delta\beta(\Delta x)^2} \qquad ...(34)$$

The inequality (34) represents the imposed condition for explicit method for Burger's- Huxley equation with diffusion term to be stable.

3.3 Stability Analysis of Crank-Nicholson Method by Fourier (Von Neumann) Method

By using the Crank-Nicholson Method for the equation (28) we obtain

$$\frac{u_{p,q+1}-u_{p,q}}{\Delta t}-\frac{\nu}{2}\left[\frac{u_{p-1,q}-2u_{p,q}+u_{p+1,q}}{(\Delta x)^2}+\frac{u_{p-1,q+1}-2u_{p,q+1}+u_{p+1,q+1}}{(\Delta x)^2}\right]=-\beta\delta u_{p,q} \qquad ...(35)$$

Substituting $u_{p,q}$ by $\psi(t)e^{i\gamma x}$ in equation (35) yields

$$\frac{\psi(t+\Delta t)e^{i\gamma x}-\psi(t)e^{i\gamma x}}{\Delta t}-\frac{\nu}{2}\left[\frac{\psi(t)e^{i\gamma(x-\Delta x)}-2\psi(t)e^{i\gamma x}+\psi(t)e^{i\gamma(x+\Delta x)}}{(\Delta x)^2}+\right.$$

$$\left.\frac{\psi(t+\Delta t)e^{i\gamma(x-\Delta x)}-2\psi(t+\Delta t)e^{i\gamma x}+\psi(t+\Delta t)e^{i\gamma(x+\Delta x)}}{(\Delta x)^2}\right]=-\delta\beta\psi(t)e^{i\gamma x}$$

$$\left[\frac{\psi(t+\Delta t)-\psi(t)}{\Delta t}\right]e^{i\gamma x}-\frac{\nu\psi(t)}{2(\Delta x)^2}[e^{-i\gamma\Delta x}-2+e^{i\gamma\Delta x}]e^{i\gamma x}-\frac{\nu\psi(t+\Delta t)}{2(\Delta x)^2}[e^{-i\gamma\Delta x}-2+e^{i\gamma\Delta x}]e^{i\gamma x}$$
$$=-\delta\beta\psi(t)e^{i\gamma x}$$

Dividing both sides in above equation by $e^{i\gamma x}$ to obtain

$$[\frac{\psi(t+\Delta t)-\psi(t)}{\Delta t}]-\frac{\nu\psi(t)}{2(\Delta x)^2}[e^{-i\gamma\Delta x}-2+e^{i\gamma\Delta x}]-\frac{\nu\psi(t+\Delta t)}{2(\Delta x)^2}[e^{-i\gamma\Delta x}-2+e^{i\gamma\Delta x}]=-\delta\beta\psi(t)$$

$$[\psi(t+\Delta t)-\psi(t)]-\frac{\nu\Delta t\psi(t)}{2(\Delta x)^2}[e^{-i\gamma\Delta x}-2+e^{i\gamma\Delta x}]-\frac{\nu\Delta t\psi(t+\Delta t)}{2(\Delta x)^2}[e^{-i\gamma\Delta x}-2+e^{i\gamma\Delta x}]=-\delta\beta\Delta t\psi(t)$$

Assume that $r=\frac{\nu\Delta t}{(\Delta x)^2}$

$$[\psi(t+\Delta t)-\psi(t)]-\frac{r\psi(t)}{2}[\cos\gamma\Delta x-i\sin\gamma\Delta x+\cos\gamma\Delta x+i\sin\gamma\Delta x-2]$$
$$-\frac{r\psi(t+\Delta t)}{2}[\cos\gamma\Delta x-i\sin\gamma\Delta x+\cos\gamma\Delta x+i\sin\gamma\Delta x-2]=-\delta\beta\Delta t\psi(t)$$

$$\psi(t+\Delta t)-\psi(t)+r\psi(t)[1-\cos\gamma\Delta x]+r\psi(t+\Delta t)[1-\cos\gamma\Delta x]=-\delta\beta\Delta t\psi(t)$$

$$\psi(t+\Delta t)-\psi(t)+r\psi(t)[2\sin^2(\gamma\Delta x/2)]+r\psi(t+\Delta t)[2\sin^2(\gamma\Delta x/2)]=-\delta\beta\Delta t\psi(t)$$

$$[1+2r\sin^2(\gamma\Delta x/2)]\psi(t+\Delta t)=[1-2r\sin^2(\gamma\Delta x/2)-\delta\beta\Delta t]\psi(t)$$

$$\frac{\psi(t+\Delta t)}{\psi(t)}=\frac{1-2r\sin^2(\gamma\Delta x/2)-\delta\beta\Delta t}{1+2r\sin^2(\gamma\Delta x/2)}$$

$$\Rightarrow\frac{\psi(t+\Delta t)}{\psi(t)}=\frac{1-[2r\sin^2(\gamma\Delta x/2)+\delta\beta\Delta t]}{1+2r\sin^2(\gamma\Delta x/2)}=\xi$$

For stability, we need $\left|\frac{\psi(t+\Delta t)}{\psi(t)}\right|\le 1$, *i.e.* $|\xi|\le 1$

$$\left|\frac{1-[2r\sin^2(\gamma\Delta x/2)+\delta\beta\Delta t]}{1+2r\sin^2(\gamma\Delta x/2)}\right|\le 1,\quad \forall r,\delta,\beta,\Delta t$$

Hence the Crank-Nicholson method for Burger's-Huxley equation is unconditionally stable.

4. Numerical result

we take $t=0.5$, $\alpha=0.1$, $\beta=0.2$, $\delta=0.3$, $s=1$, $\nu=1$, With the initial condition

$$u(x,0)=\left(\frac{\delta}{2}+\frac{\delta}{2}\tanh[A_1x]\right)^{1/s}$$

and boundary conditions

$$u(0,t)=\left(\frac{\delta}{2}+\frac{\delta}{2}\tanh[-A_1A_2t]\right)^{1/s}\text{ and }u(1,t)=\left(\frac{\delta}{2}+\frac{\delta}{2}\tanh[A_1(1-A_2t)]\right)^{1/s}$$

The Solitary wave solution of Eq. (1) is

$$u(x,t)=\left(\frac{\delta}{2}+\frac{\delta}{2}\tanh[A_1(x-A_2t)]\right)^{1/s}$$

where

$$A_1=\frac{-\alpha s+s\sqrt{\alpha^2+4\beta(1+s)}}{4(1+s)}\delta,$$

$$A_2=\frac{\delta\alpha}{1+s}-\frac{(1+s-\delta)(-\alpha+\sqrt{\alpha^2+4\beta(1+s)})}{2(1+s)}$$

Table 1: Comparison between explicit and implicit with exact solitary $\alpha=0.1,\quad \beta=0.2,\quad \delta=0.3,\quad \nu=1,\ \Delta x=0.1,\quad 0<x<1,\ t=0.5$

Explicit	Crank-Nicholson	Exact Solitary wave solution
0.1516	0.1516	**0.1516**
0.1503	0.1517	**0.1522**
0.1500	0.1520	**0.1529**
0.1502	0.1524	**0.1536**
0.1507	0.1529	**0.1542**
0.1513	0.1535	**0.1549**
0.1520	0.1542	**0.1555**
0.1528	0.1550	**0.1562**
0.1539	0.1560	**0.1568**
0.1555	0.1570	**0.1575**
0.1582	**0.1582**	**0.1582**

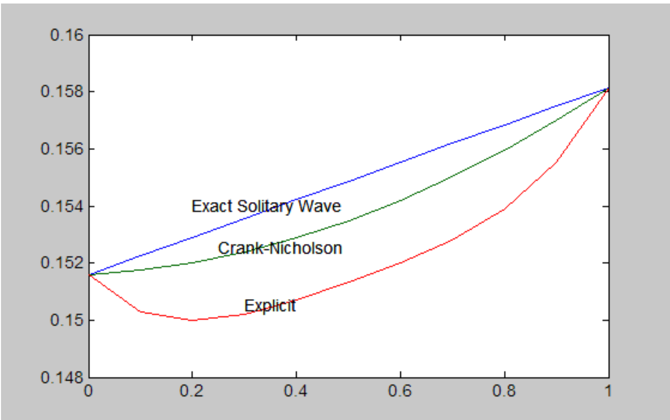

Figure (3):A comparison between explicit and Crank-Nicholson with exact solitary wave solution

It's clear from table (1) and figure (3) that Crank-Nicholson method is much more accurate and faster than explicit method to the solitary exact solution. Also, the meshes in figures (4), (5) and (6) show that Crank-Nicholson method is better to use in solving Burger's-Huxley equation. Table (2) and figure (7) show the effect of diffusion term.

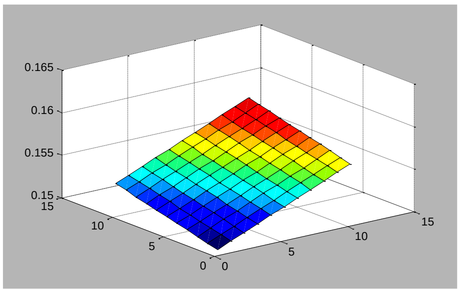

Figure (4): Solitary exact solution

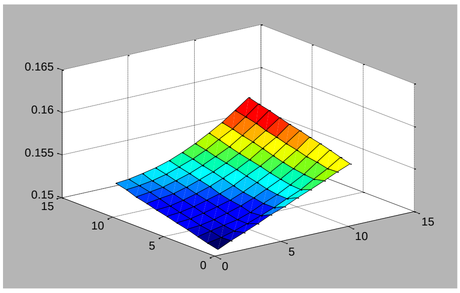

Figure (5): Crank-Nicholson Method

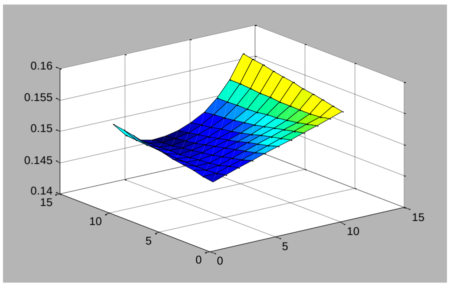

Figure (6): Explicit Method

Explicit with diffusion term	Explicit without diffusion term
0.1516	**0.1516**
0.1503	**0.1487**
0.1500	**0.1494**
0.1502	**0.1500**
0.1507	**0.1507**
0.1513	**0.1513**
0.1520	**0.1520**
0.1528	**0.1526**
0.1539	**0.1533**
0.1555	**0.1539**
0.1582	**0.1582**

Table 2 :Comparison between explicit with diffusion and without diffusion term

$\alpha = 0.1,\ \beta = 0.2,\ \delta = 0.3,\ \nu = 1, \Delta x = 0.1$

$0 < x < 1,\ t = 0.5$

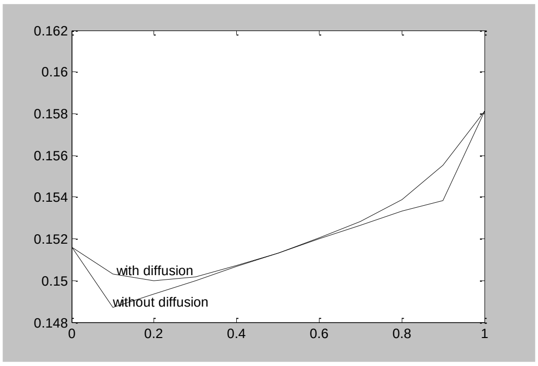

Figure (7):A comparison between with diffusion and without diffusion term

5. Conclusion

We concluded that the diffusion term to the Burger's-Huxley is as it's clear in table (2) and figure (7). Also we found that Crank-Nicholson method is more accurate and has faster convergent than explicit method as shown in table (1) and figures 3,4,5 and 6.

For the numerical stability we found that the explicit method is stable under the condition $\Delta t \le \dfrac{2(\Delta x)^2}{4\nu + \delta\beta(\Delta x)^2}$ and Crank-Nicholson method is unconditionally stable.

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