PILP-rings and fuzzy ideals

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Received on: 31/01/2007 **Accepted on: 16/04/2007**

ABSTRACT

In this paper, we study rings whose principal right ideals are left pure. Also we shall introduce the concept of a fuzzy bi-ideal in a ring, and give some properties of such fuzzy ideals. We also give a characterization of whose principal right ideal are left pure, fuzzy duo ring in terms of fuzzy deals.

Keywords: pure ideal, fuzzy ideal , duo ring,

المثاليات المضببة والحلقات من النمط PILP

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تاريخ الاستلام:2007/01/31 **تاريخ القبول: 2007/04/16**

الملخص

في هذا البحث تمت دراسة الحلقات التي يكون فيها كل مثالي أساسي أيمن هو مثالي نقي أيسر. كذلك قدمنا تصوراً عن المثاليات الثنائية المضببة. وبعض خواص المثاليات المضببة. كما أعطينا مميزات للمثاليات الأساسية اليمنى التي تكون نقياً أيسر.

الكلمات المفتاحية: المثاليات النقية, المثاليات المضببة , الحلقات من النمط duo.

1- Introduction :

Many authors have studied the theory of fuzzy rings(for example [2] and [4]). A characterization of a regular ring by fuzzy ideals due to Gupta and Kantroo [2] is very interesting. In this paper we study rings whose principal right ideal are left pure, we call such rings PILP-rings. A new properties of such rings are given and characterization of division rings in terms of rings PILP-rings are obtained . We also introduce the concepts of a fuzzy bi-ideal in a ring, and give another characterization of a ring PILP-rings.

As it is well-known, any fuzzy ideal is a fuzzy bi-ideal [3] .But in general the converse is not true, so in this paper we prove that any fuzzy bi-ideal of a semi duo, PILP-ring R is a fuzzy two-sided ideal of R.

Throughout this paper, R will denote associative ring with identity. We recall that:

1- For any element a in R, we define the right annihilator of a by
$r(a) = \{x \in R: ax = 0\}$, and likewise the left annihilator of a, $\ell(a)$.

2- A ring R is called reduced if R has no non zero nilpotent element.
3- R is called uniform if every non zero ideal of R is essential, see [5]
4- R is said to be strongly regular, if for every $a \in R$ there exists $b \in R$ such that $a= a^2 b$
5- A ring R is said to be abelian , if every idempotent element of R is central.[6]

2- PILP-Rings:

Following [1], an ideal I of a ring R is said to be a left (right) pure if for every $a \in I$, there exists $b \in I$ such that a = ba(a=ab).

Definition 2-3:

A ring R is said to be right PILP-ring if every principal right ideal is a left pure.
The following example illustrates the above definition .

Examples:

1- Let Z_6 be the ring of integers modulo 6 and I=(2) = {0,2,4}, J=(3)={0,3}. Then both (2) and (3) are pure ideals in Z_6 . Therefore Z_6 is right PILP-ring.
2- Let Z_{12} be the ring of integers modulo 12 and (2) = {0,2,4,6,8,10} is not pure ideal. Therefore Z_{12} is not right PILP –ring.

We shall begin this section with the following lemma which plays the central role in several of our proofs.

Lemma 2-2: Let R be an abelian ring, and right PILP-ring then R is a reduced ring.

Proof: Let $a \in R$ such that $a^2 = 0$. Then aR is a left pure ideal and hence there exists $b \in aR$ such that a = ba =ara for some $r \in R$. If we set e= ar , then $a= ea = ae = a^2 r =0$.
Therefore R is a reduced ring. ■

We being with the following result

Theorem 2-3: Let R be a right PILP –ring and abelian ring with $\ell(a) =0$. Then R is a division ring.

Proof : Let a be a non zero element of R such that $\ell(a) = 0$. Since R is PILP –ring, then aR is a left pure ideal and hence there exists $b \in aR$ such that a = ba =axa , for some $x \in R$. Whence $(1-ax) \in l(a) = 0$, yielding ax = 1 Now, a = axa gives $(1-xa) \in r(a) = 0$ (Lemma 2-2) , so xa = 1 . Therefore a is invertible, whence R is a division ring. ■

Our final result gives the connection between right PILP-ring and strongly regular.

Proposition 2-4: Let R be a right PILP –ring. Then the center of R is strongly regular.
Proof: Let $0 \neq a \in$ cent (R) (the centre of R). Since R is right PILP –ring. Then there exists $b \in aR$ such that $a = ba = ara$. Now, since the center of R is a belian, then applying Lemma (2-2) we have R is a reduced and this implies that r(a) = l(a) , whence $a= a^2r$. Therefore cent (R) is strongly regular.■

3- The connection between PILP-rings and fuzzy bi-ideals.

Let R be a ring. Then a function λ from R to the unit interval [0,1] is called a fuzzy subset of R. Let λ and μ be two fuzzy subset of R.
The inclusion $\lambda \subseteq \mu$ is defined by $\lambda(x) \le \mu(x)$ for all x of R, and $\lambda \cap \mu$ is defined by $(\lambda \cap \mu)(x) = \min\{\lambda(x), \mu(x)\}$, for all x of R

The following definition of the product is due to Gupta and Kantroo [2].
If λ and μ are two fuzzy subsets of R, the intrinsic product $\lambda * \mu$ is a fuzzy subset of R defined as follows. Let $x \in R$. Define

$$(\lambda * \mu)(x) = \sup_{x=\sum_{finite} a_i b_i} \min\{\lambda(a_1), \lambda(a_2), .., \lambda(a_m), \mu(b_2), ..., \mu(b_m)\}$$

if we can express $x = a_1b_1 + a_2 b_2 + ...+ a_mb_m$ for some $a_i, b_i \in R$ and for some positive integer m where each $a_i b_i \neq 0$. Otherwise, define
$(\lambda * \mu)(x) = 0$.
A fuzzy subset λ of R is called a fuzzy subring of R if
1- $\lambda(a-b) \ge \min\{\lambda(a), \lambda(b)\}$, and
2- $\lambda(ab) \ge \min\{\lambda(a), \lambda(b)\}$ for all $a,b \in R$.

A fuzzy subset λ of R is called a fuzzy left (right) ideal of R if
1- $\lambda(a-b) \ge \min\{\lambda(a), \lambda(b)\}$ and
2- $\lambda(ab) \ge \lambda(b)$ (resp. $\lambda(ab) \ge \lambda(a)$) for all $a, b \in R$.
If λ is both a fuzzy left and a fuzzy right ideal of R, then it is called a fuzzy ideal of R.

We denote by f_A the characteristic function on a subset A of R. We note that $f_R(x) = 1$ for all x of R . We shall denote this by R instead of f_R.

The following are due to Gupta and Kantroo [2]

Lemma 3-1: If λ is a fuzzy subset of a ring R, then the following conditions are equivalent .

1- $\lambda * \lambda \subseteq \lambda$

2- $\min \{\lambda(a_1), \lambda(a_2) \ldots\ldots, \lambda(a_m), \lambda(b_1), \ldots., \lambda(b_m)\}$
$\leq \lambda(a_1 b_1 + a_2 b_2 + \ldots\ldots + a_m b_m)$

Lemma 3-2: A fuzzy subset λ of a ring R is a fuzzy left (right)ideal of R if and only if

1- $\lambda(a-b) \geq \min\{\lambda(a), \lambda(b)\}$ and

2- $R * \lambda \subseteq \lambda$ (resp. $\lambda * R \subseteq \lambda$).■

A fuzzy subset λ of R is called a fuzzy bi-ideal of R if

1- $\lambda(a-b) \geq \min\{\lambda(a), \lambda(b)\}$ for all $a, b \in R$

2- $\lambda * \lambda \subseteq \lambda$ and

3- $\lambda * R * \lambda \subseteq \lambda$.

Lemma 3-3: For any non empty subset A of a ring R. Then A is a right (resp. left, two sided) ideal of R if and only if f_A is a fuzzy right (resp. left, two sided) ideal of R.

Proof : see [3]. ■

Now, we give other fuzzy characterizations of right PILP-ring.

Theorem 3-4: Let R be a PILP-ring. Then $\lambda = \lambda * R * \lambda$ for every fuzzy bi-ideal λ of R .

Proof: Let λ be any fuzzy bi-ideal of R, and a any element of R. Then since R is PILP-ring there exists an element $b \in aR$ such that
a = ba = aca for some c in R. Then we have

$$(\lambda * R * \lambda)(a) = \sup_{\substack{x=\sum x_i y_i \\ finite}} \min\{\lambda(x_i), (R * \lambda)(y_i)\}$$

$$= \geq \min\{\lambda(a), (R * \lambda)(ca)\}$$

$$= \min\{\lambda(a), \sup_{ca=\sum p_i q_i} [\min\{R(p_i), \lambda(q_i)\}]\}$$

$$\geq \min\{\lambda(a), \min\{R(c), \lambda(a)\}$$

$$= \min\{\lambda(a), \min\{1, \lambda(a)\}$$

$$\geq \min\{\lambda(a), \lambda(a)\}\} = \lambda(a)$$

and so $\lambda \subseteq \lambda * R * \lambda$. Since λ is a fuzzy bi-ideal of R , the converse inclusion holds.■

Theorem 3-5: Let R be a right PILP-ring. Then $\lambda \cap \mu = \mu * \lambda * \mu$
For every fuzzy ideal λ of R and every fuzzy bi-ideal μ of R.

Proof: Let λ and μ be any fuzzy ideal and any fuzzy bi-ideal of R, respectively. Then $\mu * \lambda * \mu \subseteq \mu * R * \mu \subseteq \mu$ and
$\mu * \lambda * \mu \subseteq R*\lambda*R \subseteq \lambda$. Thus we have $\mu * \lambda * \mu \subseteq \lambda \cap \mu$ and so $(\mu * \lambda * \mu)(a) \subseteq (\lambda \cap \mu)(a)$

Now, to see that the converse inclusion holds, Let a be any element of R. Then, since R is right PILP-ring, so a = ba = aca and a = acaca . Since λ is a fuzzy ideal of R , λ(cac) $\geq$ λ (ca) $\geq$ λ(ca) $\geq$ λ(a)

Then we have

$$(\mu * \lambda * \mu)(a) = \sup_{a=\sum_{finite} x_i y_i} \min\{\mu(x_i), (\lambda*\mu)(y_i)\}$$

$$\geq \min\{\mu(a), (\lambda * \mu)(caca)\}$$

$$= \min\{\mu(a), \sup_{caca=\sum p_i q_i} [\min\{\lambda(p_i), \mu(q_i)\}]\}$$

$$\geq \min\{\mu(a), \min\{\lambda(cac), \mu(a)\}\}$$

$$\geq \min\{\mu(a), \min\{\lambda(a), \mu(a)\}\}$$

$$= (\lambda \cap \mu)(a)$$

and so we have $\mu * \lambda * \mu \supseteq \lambda \cap \mu$

Thus we obtain that $\mu * \lambda * \mu = \lambda \cap \mu$ ■

A ring R is called fuzzy duo if every fuzzy one sided ideal of R is a fuzzy two sideds .

Recall that R is called a right semi-duo if very principal right ideal of R is a two-sided ideal generated by the same element . A left semi-duo is similarly defined. A ring R is called semi-duo if R is a right and left semi-duo.

The following theorem gives the relation between semi duo ring and fuzzy duo ring.

Theorem 3-6: For a right PILP-ring, the following conditions are equivalent.
1-R is semi-duo.
2-R is fuzzy duo.

Proof: First assume that (1) holds. Let λ be any fuzzy right ideal of R, and a and b any elements of R.Then, since the set aR is a right ideal of R it is a two-sided ideal of R by the assumption. And since R is right
PILP-ring a = ca $\in$ aRa
So ba $\in$ b (aRa) = R(aRa) = $(Ra)^2$ = $(aR)^2$ $\subseteq$ aR.
This implies that there exists an element x in R, such that ba = ax . Then , since λ is a fuzzy right ideal of R , we have λ(ba) = λ(ax) $\geq$ λ(a)

This means that λ is a fuzzy left ideal of R. It can be seen in a similar way that any fuzzy left ideal of R is a fuzzy two-sided ideal of R.
Conversely, assume that (2) holds. Let Ra be any left ideal of R.
Then it follows from Lemma (3-3) that the characteristic function f_{Ra} is a fuzzy left ideal of R. Then it is a fuzzy two-sided ideal of R by the assumption. Then it follows from Lemma (3-3) that Ra is a two-sided ideal of R. Similarly, we can see that any right ideal of R is two sided. Therefore R is semi-duo.■

The following lemmas are duo to Kuroki [3].

Lemma 3-7: For a fuzzy bi-ideal λ of a ring R,
$\lambda(axb) \geq \min\{\lambda(a), \lambda(b)\}$
holds for all a, b and x of R. ■

Lemma 3-8: Any fuzzy ideal of a ring R is a fuzzy bi-ideal of R. ■

Before closing this section, we give the following result.

Theorem 3-9: Any fuzzy bi-ideal of a right PILP semi duo ring R is a fuzzy two – sided ideal of R.

Proof: Let λ be any fuzzy bi-ideal of R and a,b any element of R. Then aR is a right ideal of R. Since R is semi-duo then it is a left ideal of R . On the other hand, since R is PILP-ring, we have
a = ca = aRa
$ba \in b(aRa) \subseteq R(aR)a \subseteq aRa$
This implies that there exists an element $x \in R$ such that ba = axa . Then since λ is a fuzzy bi-ideal of R by Lemma [3-7] we have
$\lambda(ba) = \lambda(axa) \geq \min\{\lambda(a), \lambda(a)\} = \lambda(a)$
and so λ is a fuzzy right ideal of R. It can be seen by a similar way that λ is a fuzzy right ideal of R. Therefore λ is a fuzzy two-sided ideal of R.

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