A Generalization of Von Neumann Regular Rings

Adil K. Jabbar
College of Sciences
University of Sulaimani

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ABSTRACT

In this paper, we introduce a new ring which is a generalization of Von Neumann regular rings and we call it a centrally regular ring. Several properties of this ring are proved and we have extended many properties of regular rings to centrally regular rings. Also we have determined some conditions under which regular and centrally regular rings are equivalent.

Keywords: regular rings, centrally regular rings, indecomposable, multiplicative system, Jacobson radical.

تعميم للحلقات المنتظمة من النمط فون نيومان

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الملخص

في هذا البحث قدمنا تعريفا لحلقة جديدة والتي تكون تعميما للحلقات المنتظمة من النمط فون نويمان وسميناها الحلقات المنتظمة مركزيا. وتمت البرهنة على خواص عديدة لهذه الحلقة وتمكنا من توسيع بعض خواص الحلقات المنتظمة لهذه الحلقات، وكذلك حددنا بعض الشروط التي عند توافرها تصبح الحلقات المنتظمة والحلقات المنتظمة مركزيا حلقات متكافئة.

الكلمات المفتاحية: الحلقات المنتظمة، الحلقات المنتظمة مركزيا.

Introduction:

Let R be a ring. A nonempty subset S of R is called a multiplicative system in R if $0 \notin S$ and $a,b \in S$ implies that $ab \in S$ (Larsen and McCarthy, 1971) and a multiplicative system S is called a central multiplicative system if $[S,R]=\{0\}$, where $[S,R]=\{[s,r]: s\in S, r\in R\}$ and $[s,r]=sr-rs$ (Jabbar and Majeed, 2008). If S is a central multiplicative system in R, then one can easily show that $R_S=\{a_m : a\in R, m\in S\}$ is a ring under the following operations of addition and multiplication:
(i): $a_m+b_n=(na+mb)_{mn}$ and **(ii):** $a_m b_n=(ab)_{mn}$, for all $a_m, b_n \in R_S$ (Jabbar, 2007) and this ring is known as the ring of quotients of R with respect to the central multiplicative system S or the localization of R at the central multiplicative system S, where a_m is the equivalence of (a,m) in $R\times S$

under the equivalence relation (~) defined as follows: If $(a,m),(b,t)\in R\times S$ then $(a,m)\sim(b,t)$ if and only if there exists $s\in S$ such that $s(ta-mb)=0$ (Jabbar, 2007).

A ring R is called a regular ring (Von Neumann), if for every $a\in R$, there exists $b\in R$ such that $a=aba$ (Goodearl, 1979) and an ideal I of R is called a regular ideal if, for every $a\in I$, there exists $b\in I$ such that $a=aba$ (Goodreal, 1979). The Jacobson radical of R, denoted by $J(R)$, is the intersection of all maximal ideals of R, that is, $J(R)=\bigcap_{M} M$, M is a maximal ideal of R (Larsen and McCarthy, 1971). A ring R with identity 1 is called indecomposable ring if $B(R)=\{0,1\}$ (Al-Hazmi, 2005), or equivalently, R is called indecomposable if the only non zero central idempotent element of R is the identity 1 (Burgess and Raphael, 2008).

Remarks: (Jabbar, 2007)

If R is a ring. and S is a central multiplicative system in R then:

1: For all $s\in S$, we have 0_S is the zero of R_S and $0_m=0_n$, for all $m,n\in S$. Also, for all $s\in S$, we have s_S is the identity element of R_S and it is easy to see that $m_m=n_n$ for all $m,n\in S$.

2: If $a_m,b_t\in R_S$, where $a,b\in R$ and $m,t\in S$, then $a_m=b_t$ if and only if $(a,m)\sim(b,t)$ if and only if there exists $s\in S$ such that $s(ta-mb)=0$ and $a_m=0$ if and only if there exists $u\in S$ such that $ua=0$.

3: If $r,s\in R$ and $m,n\in S$, then we have $s_n+(-s)_n=(ns-ns)_{nn}=0_{nn}=0_n$ and hence $(-s)_n=-s_n$ and now $(r+s)_m=m_m(r+s)_m=(mr+ms)_{mm}=r_m+s_m$ and $(r-s)_m=(r+(-s))_m=r_m+(-s)_m=r_m-s_m$.

4: It is necessary to mention that, if S is a central multiplicative system in R, then $[S,R]=\{0\}$ and hence $S\subseteq Z(R)$, where $Z(R)$, is the center of the ring R, that is, $sr=rs$, for all $s\in S, r\in R$.

Known Results:

The following are known results, we use them to drive our main results and one can see their proofs in the indicated references.

Lemma A: (Jabbar, 2007) Let R be a ring with identity 1 in which every non zero element of $Z(R)$ is a unit in R. If S is a central multiplicative system in R and A, B are ideals of R such that $A_S=B_S$, then $A=B$.

Theorem B: (Goodearl, 1979) Let R be a regular ring with identity 1. Then:

1: All one-sided ideals of R are idempotent, and as a consequence to this: all ideals of R are idempotent.
2: The Jacobson radical of R is zero.

Lemma C: (Jabbar, 2007) If R is a ring in which $Z(R)$ contains no proper zero divisors of R, then $Z(R)-\{0\}$ is a central multiplicative system in R.

Lemma D: (Jabbar and Majeed, 2008) Let R be a ring and S is a central multiplicative system in R. If A and B are ideals (resp. left ideals or right ideals) of R, then $A_S B_S = (AB)_S$ and $(A+B)_S = A_S + B_S$.

Lemma E: (Jabbar, 2007) Let R be a ring and S is a central multiplicative system in R. If K is a maximal ideal of R_S then there exists a maximal ideal M of R, which is disjoint from S and such that $K = M_S$.

Theorem F: (Goodearl, 1979) If R is a regular ring, then its center, $Z(R)$, is also a regular ring.

Lemma G: (Jabbar and Majeed, 2008) If R is a ring and S is a central multiplicative system in R, then $(Z(R))_S \subseteq Z(R_S)$.

Theorem H: (Goodearl, 1979) Let R be a ring with identity and J be an ideal of R. Then R is regular if and only if J and $\frac{R}{J}$ are both regular.

Theorem I: (Goodearl, 1979) A ring R is regular if and only if all ideals of R are idempotent and $\frac{R}{P}$ is regular for all prime ideals P of R.

Lemma J: (Jabbar, 2007) Let R be a ring and S is a central multiplicative system in R. If K is a prime ideal of R_S then there exists a prime ideal P of R, which is disjoint from S and such that $K = P_S$.

Theorem K: (Goodearl, 1979) Let R be a ring and let $M = \{x \in R : RxR$ is a regular ideal$\}$. Then
1: M is a regular ideal of R.
2: M contains all regular ideals of R.
3: $\frac{R}{M}$ has no non zero regular ideals.

Lemma L: (Jabbar, 2007) Let R be a ring and S is a central multiplicative system in R. If I' is an ideal of R_S, then there exists an ideal I of R such that $I' = I_S$.

Theorem M: (Tuganbaev, 2002) Let R be a ring with identity. Then the following conditions are equivalent:

1: R is regular.
2: Every principal left ideal of R is generated by an idempotent.
3: Every principal right ideal of R is generated by an idempotent.
4: Every finitely generated left ideal of R is generated by an idempotent.
5: Every finitely generated right ideal of R is generated by an idempotent.

Theorem N: (Tuganbaev, 2002)

Let R be a non zero regular ring, then R is indecomposable if and only if $Z(R)$ is a field.

The Main Results:

We mention that in all what follows R is a ring with identity unless otherwise stated.

Now it is the time to introduce the following definitions.

Definitions:

We call R a centrally regular ring if R_S is a regular ring for each central multiplicative system S in R and also we call an ideal J of R a centrally regular ideal if J_S is a regular ideal of R_S for each central multiplicative system S in R.

It is easy to prove that every regular ring is a centrally regular ring.

Theorem 1:

If R is a regular ring, then it is centrally regular and also, a regular ideal is centrally regular.

Proof:

The proof is easy ■ .

In general, a centrally regular ring may not be regular as we see in the following example.

Example:

Consider the ring $(2Z_8, +_8, \cdot_8)$, where $2Z_8 = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$. It is easy to check that this ring is not regular. On the other hand, if $2Z_8$ is not centrally regular then there exists a central multiplicative system S in $2Z_8$ such that $(2Z_8)_S$ is not regular. But the only subset of $2Z_8$ which do not contain $\bar{0}$ are the following, $\{\bar{2}\}, \{\bar{4}\}, \{\bar{6}\}, \{\bar{2}, \bar{4}\}, \{\bar{2}, \bar{6}\}, \{\bar{4}, \bar{6}\}$ and $\{\bar{2}, \bar{4}, \bar{6}\}$, so that S must be one of these sets. By simple computations one can easily see that non of these sets is a multiplicative system in $2Z_8$ which is a contradiction. Hence $2Z_8$ is a centrally regular ring which is not regular.

Lemma 2:

If every non zero element of $Z(R)$ is a unit in R, S is a central multiplicative system in R and A, B are ideals (resp. left ideals or right ideals) of R such that $A_S = B_S$ then $A = B$.

Proof:

One can use the same argument as in the proof of **Lemma A**, and getting the result ■ .

Lemma 3:

If every nonzero element of $Z(R)$ is a unit in R, then $Z(R)-\{0\}$ contains no proper zero divisors of R.

Proof:

The proof is easy ■ .

Now we give a condition under which the properties of regular rings given in **Theorem B** and **Theorem F** can be extended to centrally regular rings.

Theorem 4:

Let R be a centrally regular ring in which every non zero element of $Z(R)$ is a unit in R. Then show that:

1: All one-sided ideals of R are idempotent.
2: All ideals of R are idempotent.
3: The Jacobson radical of R is zero, that is, $J(R) = 0$.
4: $Z(R)$ is a regular ring.

Proof:

1: By **Lemma 3**, we have $Z(R)-\{0\}$ contains no proper zero divisors of R and by **Lemma C**, we have $Z(R)-\{0\}$ is a central multiplicative system in R. If we put $S = Z(R)-\{0\}$, then since R is a centrally regular ring, so R_S is a regular ring. Now let A be any left ideal of R. It is easy to check that A_S is a left ideal of the regular ring R_S. Hence by **Theorem B**, we get A_S is idempotent, that is, $(A_S)^2 = A_S$. But, then from **Lemma D**, we get $(A_S)^2 = (A^2)_S$. Hence we get $(A^2)_S = A_S$. Then by **Lemma 2**, we get $A^2 = A$. So that, A is idempotent and if A is a right ideal, then by the same argument we can show that A is again idempotent ■ .

2: The proof is as the same argument as in the proof of (**1**) ■ .

3: From **Lemma 3** and **Lemma C**, we have $Z(R)-\{0\}$ is a central multiplicative system in R, so let $S=Z(R)-\{0\}$ and since R is centrally regular, so R_S is a regular ring and hence by **Theorem B**, we get $J(R_S)=0$. Next, to show $(J(R))_S \subseteq J(R_S)$. Let $a_S \in (J(R))_S$, where $a \in J(R)$ and $s \in S$. If K is any maximal ideal of R_S, then by **Lemma E**, there exists a maximal ideal M of R such that $M \cap S = \phi$ and $K = M_S$. Since $a \in J(R)$, so $a \in M$ and thus $a_s \in M_S = K$. Hence $a_s \in J(R_S)$, which implies that $(J(R))_S \subseteq J(R_S)$. Thus we get $(J(R))_S = 0$. Finally, by using the result of **Lemma 2,** we get $J(R)=0$ ■ .

4: By **Lemma 3** and **Lemma C**, we have $S=Z(R)-\{0\}$ is a central multiplicative system in R and as R is centrally regular, so that R_S is a regular ring. Hence by **Theorem F**, $Z(R_S)$ is a regular ring. Now to show $Z(R)$ is regular. Let $a \in Z(R)$. Since $1 \in S$, so $a_1 \in (Z(R))_S$. Using **Lemma G**, we get $a_1 \in Z(R_S)$ and thus, there exists $b_t \in Z(R_S)$, where $b \in R, t \in S$ such that $a_1 = a_1 b_t a_1 = (aba)_{1t1} = (aba)_t$. Then there exists $s \in S$ such that $sta = saba$. Since $s,t \in S$, so they are non zero elements of $Z(R)$ and hence they are units in R, thus $s^{-1}, t^{-1} \in R$. Next to show that $t^{-1}, b \in Z(R)$. Since S is central, so for all $r \in R$, we have $tr = rt$. Then $t^{-1}trt^{-1} = t^{-1}rtt^{-1}$. Hence $rt^{-1} = t^{-1}r$, so that $t^{-1} \in Z(R)$. To show $b \in Z(R)$. Since $b_t \in Z(R_S)$, so for all $r \in R$, we have $(br)_{tt} = b_t r_t = r_t b_t = (rb)_{tt}$. Hence there exists $u \in S$ such that $utt(br-rb)=0$. Then, since $u,t \in S$, we get that u,t are non zero elements of $Z(R)$ and thus $u^{-1}, t^{-1} \in R$. Then $br - rb = t^{-1}t^{-1}u^{-1}utt(br-rb) = t^{-1}t^{-1}u^{-1}0 = 0$. So that $br = rb$, for all $r \in R$ and thus $b \in Z(R)$. Hence $t^{-1}b \in Z(R)$. Since S is central, so we get $a = t^{-1}s^{-1}sta = t^{-1}s^{-1}saba = at^{-1}ba$, where $t^{-1}b \in Z(R)$. Hence $Z(R)$ is a regular ring ■ .

Next we prove the following result which determines the relation between the regularity of ideals in the both rings R and R_S.

Lemma 5:

If every non zero element of $Z(R)$ is a unit in R, S is a central multiplicative system in R and if J is an ideal of R, then J is a regular ideal of R if and only if J_S is a regular ideal of R_S.

Proof:

Let J be a regular ideal of R. By using the same argument as in **Theorem 1**, we can show that J_S is a regular ideal of R_S.

Conversely, let J_S be a regular ideal of R_S. To show J is a regular ideal of R. Let $a \in J$. Then as $S \neq \phi$, there exists an element $s \in S$ such that $a_S \in J_S$. Hence there exists $b_t \in J_S$, for $b \in J$ and $t \in S$, such that $a_S = a_S b_t a_S = (aba)_{sts}$. Then there exists $u \in S$, such that $ustsa = usaba$. Since u, s, t are all non zero elements of $Z(R)$, so $u^{-1}, s^{-1} t^{-1} \in R$, and then one can easily get that $a = a(s^{-1}t^{-1}b)a$, where $s^{-1}t^{-1}b \in J$. Hence J is a regular ideal of R ■ .

Now, with the aid of the last lemma we can extend the result of **Theorem H**, to centrally regular rings.

<u>Theorem 6:</u>

If every non zero element of $Z(R)$ is a unit in R and J is an ideal of R, then R is centrally regular if and only if both J and $\frac{R}{J}$ are regular.

<u>Proof:</u>

Let R be centrally regular. To show J and $\frac{R}{J}$ are regular. By **Lemma 3** and **Lemma C**, we have $S = Z(R) - \{0\}$ is a central multiplicative system in R, so that R_S is a regular ring. Then by **Theorem H**, we have J_S and $\frac{R_S}{J_S}$ both are regular . Hence by **Lemma 5**, we get J is a regular ideal of R. Next, let $r + J \in \frac{R}{J}$ be any element, where $r \in R$. Then as $1 \in S$ we get $r_1 \in R_S$ and hence $r_1 + J_S \in \frac{R_S}{J_S}$, so there exists $a_S + J_S \in \frac{R_S}{J_S}$, where $a \in R, s \in S$, such that

$r_1 + J_S = (r_1 + J_S)(a_S + J_S)(r_1 + J_S) = (r_1 a_S r_1) + J_S = (rar)_S + J_S$. Hence

$r_1 - (rar)_S \in J_S$. Then we get

$(rs - rar)_S = (rs)_S - (rar)_S = r_1 s_S - (rar)_S = r_1 - (rar)_S \in J_S$. Hence

$(rs - rar)_S = b_t$, for some $b \in J, t \in S$ and thus, there exists $u \in S$, such that $ut(rs - rar) = usb$. Since S is central, so we get $utsr - utrar = usb$. But since $u, t, s \in S$, so u, t, s are nonzero elements of $Z(R)$ and hence they are units of R, that is, $u^{-1}, t^{-1}, s^{-1} \in R$. Then from the last equation we get $r - rs^{-1}ar = t^{-1}b \in J$.

Put $c = s^{-1}a$, so we get $r - rcr \in J$, where $c = s^{-1}a \in R$. Thus $r + J = rcr + J = (r + J)(c + J)(r + J)$, where $c + J \in \frac{R}{J}$. Hence $\frac{R}{J}$ is regular.

Conversely, suppose that both J and $\frac{R}{J}$ are regular. Then, by **Theorem H**, we get R is a regular ring and then by **Theorem 1**, we get R is centrally regular ■ .

Next, we give a condition which makes both rings R and R_S as indecomposable rings.

Theorem 7:

If R is non zero and $Z(R)$ contains no proper zero divisors of R with S is a central multiplicative system in R, then:

1: R is an indecomposable ring.

2: R_S is indecomposable ring.

Proof:

1: First, we will show that R is indecomposable, so let a be any non zero central idempotent element of R. This means that a is a non zero element of $Z(R)$.

Since a is idempotent, so $a^2 = a$, that is, $a(a-1) = 0$. If $a - 1 \neq 0$, then a is a proper zero divisor of R, which is a contradiction (since $Z(R)$ contains no proper zero divisors of R) and thus $a - 1 = 0$, that is, $a = 1 =$ the identity of R
and thus R is indecomposable ■ .

2: Next, to show R_S is indecomposable. Let a_S be a non zero central idempotent element of R_S, where $a \in R, s \in S$. If $a = 0$, then $a_S = 0$, which is a contradiction, so we get $a \neq 0$. Next to show $a \in Z(R)$. Let $r \in R$. Then, since a_S is a central element of R_S, so $a_S \in Z(R_S)$ and as $r_S \in R_S$, we get $a_S r_S = r_S a_S$. Hence $(ar)_{SS} = (ra)_{SS}$, so there exists $t \in S$ such that $tss(ar - ra) = 0$. As $Z(R)$ contains no proper zero divisors of R, we get $ar = ra$, so that $a \in Z(R)$. That means a is a non zero element of $Z(R)$. Now, since a_S is idempotent element of R_S, so we have $(a_s)^2 = a_s$. Hence $a_S a_S = a_S$, which implies that $(a^2)_{ss} = a_s = a_s s_s = (as)_{ss}$. By the same steps, as in the above, we obtain $a^2 = as$, that is $a(a - s) = 0$. Then, if $a - s \neq 0$, then a is a proper zero divisor of R, that means, $Z(R)$ contains the proper zero divisor a which is a contradiction. Hence $a - s = 0$, and thus $a = s$, then $a_S = s_S =$ the identity of

R_S. So that R_S is indecomposable ■ .

It is known that, a field contains no proper zero divisors, but the converse is not true, in general, that is, if $Z(R)$ is a field, then it has no proper zero divisors and now, as a corollary to **Theorem 7**, we prove that the converse of the above statement is true also when the ring is a non zero regular ring, that is, if R is a non zero regular ring and $Z(R)$ contains no proper zero divisors, then $Z(R)$ is a field.

Corollary 8:

If R is non zero and regular, for which $Z(R)$ contains no proper zero divisors of R, then $Z(R)$ is a field.

Proof:

By **Theorem 7**, we get R is indecomposable and by **Theorem N**, we get that
$Z(R)$ is a field ■ .

Next, we generalize the result of **Theorem I**, to centrally regular rings and as follows:

Theorem 9:

If every non zero element of $Z(R)$ is a unit in R. Then R is centrally regular if and only if every ideal of R is idempotent and $\frac{R}{P}$ is a regular ring, for every prime ideal P of R.

Proof:

Let R be centrally regular. From **Theorem 4**, we get that every ideal of R is idempotent. Now let P be any prime ideal of R, then from **Theorem 6**, we get $\frac{R}{P}$ is regular.

Conversely, suppose that every ideal of R is idempotent and $\frac{R}{P}$ is regular for every prime ideal P of R. To show R is centrally regular. Let S be any central multiplicative system in R. It is required to show that R_S is regular. If I' is any ideal of R_S, then from **Lemma L** , there exists an ideal I of R such that $I' = I_S$. Hence by the given condition we get I is idempotent and then by using **Lemma D**, we get $(I')^2 = I'I' = I_S I_S = (I^2)_S = I_S = I'$. That is, every ideal of R_S is idempotent. Let P' be any prime ideal of R_S. To show $\frac{R_S}{P'}$ is regular. By **Lemma J**, there exists a prime ideal P of R such that $P \cap S = \phi$ and $P' = P_S$. So that by the given condition $\frac{R}{P}$ is regular. Now let $r_S + P_S \in \frac{R_S}{P_S}$, where $r \in R, s \in S$. Then $r + P \in \frac{R}{P}$. As $\frac{R}{P}$ is regular, there exists $a + P \in \frac{R}{P}$, for some $a \in R$, such that $r + P = (r + P)(a + P)(r + P) = rar + P$. Hence $r - rar \in P$. So that $r_S - (rar)_S = (r - rar)_S \in P_S$. Hence we get $r_S + P_S = (rar)_S + P_S =$ $(rar)_S s_S s_S + P_S = (rarss)_{SSS} + P_S = (rassr)_{SSS} + P_S = r_S (ass)_S r_S + P_S =$ $(r_S + P_S)((ass)_S + P_S)(r_S + P_S)$, where $(ass)_S + P_S \in \frac{R_S}{P_S}$. Thus we get $\frac{R_S}{P_S}$ is regular, that is $\frac{R_S}{P'}$ is regular for all prime ideals P' of R_S. Hence by **Theorem I**, we get R_S is regular and thus R is a centrally regular ring ■ .

<u>Theorem 10:</u>

If every non zero element of $Z(R)$ is a unit in R and $x \in R$. Then RxR is a regular ideal of R if and only if RxR is a centrally regular ideal of R.

<u>Proof:</u>

Let RxR be a regular ideal of R. Then, RxR is a centrally regular ideal of R.

Conversely, let RxR be a centrally regular ideal of R. To show RxR is a regular ideal R. By **Lemma 3** and **Lemma C**, $S = Z(R) - \{0\}$ is a central multiplicative system in R and thus $(RxR)_S$ is a regular ideal of R_S. Now let $axb \in RxR$ be any element, where $a, b \in R$. As $1 \in S$, we get $(axb)_1 \in (RxR)_S$ and hence there exists $u, v \in R, t \in S$ such that $(axb)_1 = (axb)_1 (uxv)_t (axb)_1 =$ $(axbuxvaxb)_t$. Hence there exists $s \in S$ such that $st(axb) = s(axbuxvaxb)$. Using the fact that S is central and that non zero elements of $Z(R)$ are units in R, so by simple computations we get $axb = axb(t^{-1}uxv)axb$, where $t^{-1}uxv \in RxR$. Hence RxR is a regular ideal of R ■ .

By using the result of **Theorem 10**, we can extend the result of **Theorem K**, to
centrally regular rings and as follows:

<u>Theorem 11:</u>

If every non zero element of $Z(R)$ is a unit in R and let $M = \{x \in R : RxR$ is a centrally regular ideal of $R\}$. Then:

1: M is a regular ideal of R.

2: M contains all regular ideals of R.

3: $\frac{R}{M}$ has no non zero regular ideals.

<u>Proof:</u>

By applying **Theorem 10**, we can see that $M=\{x\in R: RxR$ is a centrally regular ideal of $R\}=\{x\in R: RxR$ is a regular ideal of $R\}$ and now by applying **Theorem K**, the proof will follow at once ■ .

Now we prove the following lemma, which will be used in driving our last result.

<u>Lemma 12:</u>

If S is a central multiplicative system in R with $1\in S$, then $(Rx)_S=R_S x_s=R_S x_1$ and $(xR)_S=x_s R_S=x_1 R_S$, for all $x\in R$ and for all $s\in S$.

<u>Proof:</u>

Let $x\in R$ and $s\in S$. To show $(Rx)_S=R_S x_s$. Let $(rx)_t\in(Rx)_S$, where $r\in R, t\in S$. Then $(rx)_t=(rx)_t s_s=(rs)_t x_s\in R_S x_s$, and thus $(Rx)_S\subseteq R_S x_s$ and if $r_t x_s\in R_S x_s$, for $r\in R, t\in S$, then $r_t x_s=(rx)_{ts}\in(Rx)_S$, so that $R_S x_s\subseteq(Rx)_S$. Hence $(Rx)_S=R_S x_s$. Next, to show that $(Rx)_S=R_S x_1$. Since $(rx)_s=r_s x_1$, for all $r\in R$, so we get $(Rx)_S=R_S x_1$. Hence we get $(Rx)_S=R_S x_s=R_S x_1$.

The proof of the second part is as the same steps of the proof of the first part ■ .

Finally, we generalize the result of **Theorem M**, to centrally regular rings and as follows:

<u>Theorem 13:</u>

If every non zero element of $Z(R)$ is a unit in R. Then the following conditions are equivalent:

1: R is a centrally regular ring.
2: Every principal left ideal of R is generated by an idempotent.
3: Every principal right ideal of R is generated by an idempotent.
4: Every finitely generated left ideal of R is generated by an idempotent.
5: Every finitely generated right ideal of R is generated by an idempotent.

<u>Proof:</u>

$(1\leftrightarrow 2)$: First, let R be a centrally regular ring. By **Lemma 3** and **Lemma C**, we get $S=Z(R)-\{0\}$ is a central multiplicative system in R and so that R_S is a regular ring. Let Rx, be any principal left ideal of R, where $x\in R$. It is easy to show that $(Rx)_S$ is a left ideal of R_S. From **Lemma 12**, we have

$(Rx)_S = R_S x_1$. Hence $(Rx)_S$ is a principal left ideal of the regular ring R_S, so by **Theorem M**, we get $(Rx)_S$ is generated by an idempotent element of R_S, say e_S, that is, $(Rx)_S = R_S e_S$, where e_S is an idempotent element of R_S. Then we have $(e^2)_{ss} = (ee)_{ss} = e_s e_s = e_s = s_s e_s = (se)_{ss}$ and since s is central and non zero elements of $Z(R)$ are units in R, we get $e^2 = se$. Since s is a non zero element of $Z(R)$, so it is a unit in R. Hence there exists $u \in R$ such that $us = 1 = su$ (Note that $u^{-1} = s$ and $s^{-1} = u$). Next, we will show $u \in Z(R)$. Let $r \in R$, then since $s \in S$, so $s \in Z(R)$. Hence $rs = sr$. Then we get $ursu = usru$, that is, $ur = ru$, for all $r \in R$, so that $u \in Z(R)$. Then, we get $(ue)^2 = ueue = uuee = uue^2 = uuse = ue$ and thus ue is an idempotent element of R. To show $Rx = R(ue)$. Let $b \in Rx$. Then, as $1 \in S$, we have $b_1 \in (Rx)_S = R_S e_S$, so that, $b_1 = k_t e_s = (ke)_{ts}$, for some $k \in R, t \in S$. Then, there exists $v \in S$, such that $vtsb = vke$. As $v, t \in S$, they are units in R. So that, $v^{-1}, t^{-1} \in R$. Then since $u \in Z(R)$, so we have $b = s^{-1}t^{-1}v^{-1}vtsb = s^{-1}t^{-1}v^{-1}vke = ut^{-1}ke = t^{-1}kue \in R(ue)$. Thus $Rx \subseteq R(ue)$. Next, we will show $e \in Rx$. Since $(Rx)_S = R_S e_S$, so we get $e_s = s_s e_s \in R_S e_s = (Rx)_S$. Hence, there exists $c \in R, m \in S$ such that $e_s = (cx)_m$. Then, there exists $n \in S$ such that $nme = nscx$. Since $n, m \in S$, so they are non zero elements of $Z(R)$ and hence they are units in R, so that $n^{-1}, m^{-1} \in R$. Then we get $e = m^{-1}n^{-1}nme = m^{-1}n^{-1}nscx = m^{-1}scx \in Rx$. Then, since Rx is a left ideal of R, so we have $r(ue) = rue \in Rx$, for all $r \in R$, which means that $R(ue) \subseteq Rx$. Hence $Rx = R(ue)$, that means Rx is generated by the idempotent element ue of R.

Conversely, suppose that every principal left ideal of R is generated by an idempotent. Then, by **Theorem M**, we have R is a regular ring and then by **Theorem 1**, we get R is a centrally regular ring.

$(1 \leftrightarrow 3)$ We can proceed exactly by the same way as in the previous proof just by taking xR as a principal right ideal of R and getting the result.

$(1 \leftrightarrow 4)$ Let R be a centrally regular ring. Again by **Lemma 3** and **Lemma C**, we get $S = Z(R) - \{0\}$ is a central multiplicative system in R and so that R_S is a regular ring. Let $Rx_1 + Rx_2 + ... + Rx_n$ be any finitely generated left ideal of R, where $x_1, x_2, ..., x_n \in R$. Using **Lemma D** and **Lemma 12**, we get $(Rx_1 + Rx_2 + ... + Rx_n)_S = (Rx_1)_S + (Rx_2)_S + ... + (Rx_n)_S =$ $R_S(x_1)_1 + R_S(x_2)_1 + ... + R_S(x_n)_1$. That means, $(Rx_1 + Rx_2 + ... + Rx_n)_S$ is a finitely generated left ideal of the regular ring R_S and thus by **Theorem M**, we get that $(Rx_1 + Rx_2 + ... + Rx_n)_S$ is generated by an idempotent element of R_S, say e_S. So that $(Rx_1 + Rx_2 + ... + Rx_n)_S = R_S e_S$, where e_S is an idempotent element of R_S. Then, since e_S is idempotent, so as the same steps as in the above, there exists $u \in Z(R)$ such that $us = 1 = su$, with $u = s^{-1}, s = u^{-1}$ and ue is an idempotent element of R. To show $Rx_1 + Rx_2 + ... + Rx_n = R(ue)$. Let $b \in Rx_1 + Rx_2 + ... + Rx_n$. Then, as $1 \in S$, we have $b_1 \in (Rx_1 + Rx_2 + ... + Rx_n)_S = R_S e_S$, so there exists $a \in R, k \in S$, such that $b_1 = a_k e_S = (ae)_{ks}$. Then, there exists $v \in S$, such that $vksb = vae$. As $v, k \in S$, they are units in R. So that, $v^{-1}, k^{-1} \in R$. Then since $u \in Z(R)$, so we have $b = s^{-1}k^{-1}v^{-1}vksb = s^{-1}k^{-1}v^{-1}vae = uk^{-1}ae = k^{-1}aue \in R(ue)$. Thus we get $Rx_1 + Rx_2 + ... + Rx_n \subseteq R(ue)$. Next, we will show $e \in Rx_1 + Rx_2 + ... + Rx_n$. Now, we have $e_S = s_S e_S \in R_S e_S = (Rx_1 + Rx_2 + ... + Rx_n)_S$, so there exists $c_1, c_2, ..., c_n \in R, l \in S$ such that $e_S = (c_1x_1 + c_2x_2 + ... + c_nx_n)_l$. Then, there exists $n \in S$ such that $nle = ns(c_1x_1 + c_2x_2 + ... + c_nx_n)$. Since $n, l \in S$, so they are non zero elements of $Z(R)$ and hence they are units in R, so that $n^{-1}, l^{-1} \in R$.

Then we get $e = l^{-1}n^{-1}nle = l^{-1}n^{-1}ns(c_1x_1 + c_2x_2 + ... + c_nx_n) =$ $l^{-1}sc_1x_1 + l^{-1}sc_2x_2 + ... + l^{-1}sc_nx_n \in Rx_1 + Rx_2 + ... + Rx_n$. Then, since $Rx_1 + Rx_2 + ... + Rx_n$ is a left ideal of R, so we have $r(ue) = rue \in Rx_1 + Rx_2 + ... + Rx_n$ for all $r \in R$, which means that $R(ue) \subseteq Rx_1 + Rx_2 + ... + Rx_n$. Hence $Rx_1 + Rx_2 + ... + Rx_n = R(ue)$, that means, $Rx_1 + Rx_2 + ... + Rx_n$ is generated by the idempotent element ue of R.

Conversely, suppose that every finitely generated left ideal of R is generated by an idempotent. Then, by **Theorem M**, we have R is a regular ring and then by **Theorem 1**, we get R is a centrally regular ring.

$(1 \leftrightarrow 5)$ We can proceed exactly by the same way as in the previous proof just by taking $x_1R + x_2R + ... + x_nR$ as a finitely generated right ideal of R and getting the result ■ .

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