On Rings With Types Of γ(n)-Regularity

Abdul Aali Jasim Mohammad
College of Education-University of Mosul

Received: 23/9/2010 ; Accepted: 24/11/2010

Abstract:

Let R be an associative ring with identity . For a fixed integer n >1 , an element a in R is said to be γ(n)-regular (γ(n)-strongly regular) if there exists b in R such that $a=ab^na$ ($a=a^2b^n$) . So a ring R is said to be γ(n)-regular (γ(n)-strongly regular) for a positive integer n $\neq$1 , if every element of R is γ(n)-regular (γ(n)-strongly regular) .

In this paper we investigate some characterizations and several basic properties of those rings , also the connection between them and rings of some kind of commutivity .

حلقة منتظمة من النمط γ-(n)

عبد العالي جاسم محمد
كلية التربية- جامعة الموصل

ملخص البحث :

لتكن R حلقة تجميعية بمحايد . لعدد صحيح n اكبر من واحد ، العنصر a في R يسمى منتظما من النمط γ-(n) (منتظما بقوة من النمط γ-(n)) اذا وجد عنصر مثل b في R بحيث ان $a=ab^na$ ($a=a^2b^n$) . وهكذا فان الحلقة R تسمى حلقة منتظمة من النمط γ-(n) (حلقة منتظمة بقوة من النمط γ-(n)) حيث ان n عدد صحيح اكبر من واحد، اذا كان كل عنصر من R هو منتظما من النمط γ-(n) (منتظما بقوة من النمط γ-(n)) .

في هذا البحث درسنا بعض الصفات الاساسية لهكذا حلقات و كذلك الروابط بينها وبين حلقات ذات صفات مشابهة للابدالية .

0 : Introduction :

The conception of Von Neumann regular rings occurred in 1936 [13] defined a regular ring as a ring R with property that for each $a\in R$ there exists $b\in R$ such that $a=aba$.

A ring R with property that for each $a\in R$ there exists $b\in R$ such that $a=a^2b$, is called strongly regular ring . This concept has been defined some sixty year ago by R.F.Arens and I.Kaplansky [1] and was studied in recent years by many others (cf. [2] , [8] , [9]) . Note that in such rings $a=ba^2$ if and only if $a=a^2b$ [4] .

As a generalization of regularity property , MecCoy [6] defined π-regular rings , that is a ring R with every $a\in R$, a^n is regular element for some positive integer n .(cf. [6] , [9] , [11] , [12]) .In 1954 Azumaya [2] defined strongly π-regular rings , that is ring R such that for every $a\in R$, there exists $b\in R$ and a positive integer in such that $a^n=a^{n+1}b$. Also Ehrlich [5] defined unit regular ring , that is a ring R with every $a\in R$ there exists a unit $u\in R$ such that $a=aua$.

A ring R is said to be γ-regular if for every $a\in R$ there exists $b\in R$ and an integer $n>1$ such that $a=ab^na$. This concept was introduced in 2006 by A.J.Mohammad and S.M.Salih [10] . Similarly definition of γ-strongly regular rings .

A ring R is said to be reduced , if R contains no non-zero nilpotent elements . So every idempotent element in a reduced ring is central [9] .

For $a\in R$, r(a) and l(a) denoted the right and left annihilator of a respectively .

Finally , throughout this paper R is associative ring with identity .

§1 : γ(n)-regular rings :

In this section we introduce the definition of γ(n)-regular rings as special case of regular rings .

Definition 1.1 :

An element a of a ring R is called γ(n)-regular (for an integer n>1) if $a=ab^na$ for some $b\in R$

A ring R is said to be γ(n)-regular ring if every element of R is γ(n)-regular .

Hence every γ(n)-regular ring is γ-regular ring and every γ-regular ring is a regular ring . So every γ(n)-regular ring is regular ring .

Examples 1.2 :

(1) The ring Z_3 is γ(3)-regular ring .

(2) Boolean rings are γ(n)-regular rings for each integer n>1 .

Note that every γ(n)-regular ring is γ-regular ring , but the converse is not true in general , for example Z_5 is γ-regular ring but not γ(n)-regular ring , and the ring Q of rational numbers is regular but not γ(n)-regular.

It is clear that a homomorphic image of γ(n)-regular ring is γ(n)-regular ring .

Proposition 1.3 :

Let R be a γ(n)-regular ring then for every $a\in R$, l(a) and r(a) are direct summands .

Proof :

Since R is γ(n)-regular ring , then for each $a \in R$ there exists $b \in R$ such that $a=ab^n a$. So $(ab^n-1)a=0$. That is $ab^n-1 \in l(a)$. So $1-ab^n \in l(a)$. Since $1=ab^n+(1-ab^n)$, then $R=(ab^n)R+l(a)$.

Now let $x \in (Rab^n) \cap l(a)$, then $x \in (Rab^n)$ and $xa=cab^n a = 0$. That is $x=cab^n$ for some $c \in R$. Hence $xa=cab^n a$, then $ca=0$. So $cab^n=0$. Hence $x=0$ there for l(a) is direct summand .

Similarly , r(a) is direct summand . ■

Proposition 1.4 :

If R is a ring with every nonzero element $a \in R$ there is a unique element $b \in R$ such that $a^n=a^n ba^n$ where n is an integer greater than one , then b is γ(n)-regular element .

Proof :

Since $a^n=a^n ba^n$ for each $a \in R$, then R has no divisors of zero . then by cancellation law , $1=a^n b$. So , $b=ba^n b$.

Therefore b is γ(n)-regular element . ■

Theorem 1.5 :

Let I be an ideal of a γ(n)-regular ring R , then the ring R/I is also γ(n)-regular ring .

Proof :

Let $a+I \in R/I$, then $a \in R$. Since R is γ(n)-regular ring then there exists $b \in R$ such that $a=ab^n a$. So $a+I=ab^n a+I=(a+I)(b+I)^n(a+I)$.

Therefore R/I is γ(n)-regular ring . ■

Definition 1.6 :

An ideal I of a ring R is said to be γ(n)-regular ideal if for every element $a \in I$ there exists $b \in I$ such that $a = ab^n a$.

Lemma 1.7 :

Let x and y be two elements of aring R such that , $xy^t x - x$ is γ(n)-regular element , with $1 \neq t \in Z^+$, then x is regular element .

Proof :

Since $xy^t x - x$ is γ(n)-regular element , then there exists an element $b \in R$ such that

$xy^t x - x = (xy^t x - x)b^n(xy^t x - x)$. Hence

$xy^t x - x = xy^t xb^n xy^t x - xy^t xb^n x - xb^n xy^t x + xb^n x$.

So $x = xy^t x - xy^t xb^n xy^t x + xy^t xb^n x + xb^n xy^t x - xb^n x$

$= x(y^t - y^t xb^n xy^t + y^t xb^n + b^n xy^t - b^n)x$.

Therefore x is a regular element . ■

Theorem 1.8 :

Let I be a γ(n)-regular ideal of a ring R and R/I is also γ(n)-regular ring . Then R is regular ring .

Proof :

Let $a \in R$. So $a + I \in R/I$. Since R/I is γ(n)-regular ring , then there exists $b + I \in R/I$ such that $a+I=(a+I)(b+I)^n(a+I)$. Hence $a+I=(ab^n a)+I$. So $a-ab^n a \in I$. That is , $a-ab^n a$ is γ(n)-regular element . Then a is regular element [Lemma 1.7] .

Therefore R is a regular ring . ■

Corollary 1.9 :

Let R_1 and R_2 be two γ(n)-regular rings . Then $R_1 \oplus R_2$ is regular ring .

Proof :

Since $(R_1 \oplus R_2)/R_1 \cong R_2$ and R_2 is γ(n)-regular ring , then $R_1 \oplus R_2$ is regular ring [Theorem 1.8] . ■

Theorem 1.10 :

A finite direct sum of γ(n)-regular rings is a regular ring .

Proof :

The proof is by mathematical induction ■

Theorem 1.11 :

Every ideal in a reduced γ(n)-regular ring is γ(n)-regular ideal .

Proof :

Let I be an ideal of a reduced γ(n)-regular ring R , and $a \in I$. Then there exists $b \in R$ such that $a = ab^n a$. Since $b^n a$ is idempotent element , it is central .

So , $(b^{n+1}a)(b^{n+1}a) = (b^{n+1}a)(bb^n a) = (b^{n+1}a)(b^n ab)$
$= b^{n+1}(ab^n a)b = (b^{n+1}a)b$.

Now let $y = b^{n+1}a$, then $y \in I$.

So , $ay^n a = a(b^{n+1}a)a = a[(b^{n+1}a)(b^{n+1}a) \ldots (b^{n+1}a)(b^{n+1}a)]a = a[(b^{n+1}a)b \ldots bb]a = a[(b^{n+1}a)b^{n-1}]a = a[(bb^n a)b^{n-1}]a = a(b^n abb^{n-1})a = a(b^n ab^n)a = (ab^n a)(b^n a) = ab^n a = a$.

Therefore I is γ(n)-regular . ■

2 :Strongly γ(n)-regular rings :

In this section we introduce the definition of strongly γ(n)-regular rings and we will show that this definition is equivalence to the definition of γ(n)-regular rings where the rings are commutative , reduced rings , and so on .

Definition 2.1 :

A ring R is called strongly γ(n)-regular ring (for some integer $n>1$) if for every element $a\in R$, there exists $b\in R$ such that $a=a^2b^n$.

Remark 2.2 :

(1) In strongly γ(n)-regular ring R , $a=a^2b^n$ if and only if $a=b^na^2$ [4] .

(2) It is obvious that a homomorphic image of a strongly γ(n)-regular ring is strongly γ(n)-regular ring .

Theorem 2.3 :

For any ring R , the following statements are equivalent :

(1) R is strongly γ(n)-regular ring .

(2) R is reduced γ(n)-regular ring .

Proof :

$1 \Longrightarrow 2$. Since R is strongly γ(n)-regular ring , then for every $a\in R$ there exists $b\in R$ such that $a=a^2b^n=b^na^2$. So $b^na = b^n(a^2b^n) = (b^na^2)b^n = ab^n$. Hence $ab^na = a^2b^n = a$. That is R is γ(n)-regular ring .

Now suppose that $c^t=0$ for some $c\in R$ and t is a positive integer . Since $c=c^2d^n$ for some $d\in R$. Then $c^{t-1} = c^{t-2}c = c^{t-}$

${}^2c^2d^n = c^td^n = 0$. Similarly $c^{t-2}=0$ and so on , to have $c=0$. Hence R is a reduced ring .

2 $\Longrightarrow$ 1 . Since R is γ(n)-regular ring then for every $a\in R$ there exists $b\in R$ such that $a=ab^na$. So , $(a-a^2b^n)^2 = a^2-a^3b^n-a^2b^na+a^2b^na^2b^n = a^2-a^3b^n-a^2+a^3b^n = 0$. Since R is reduced then $a-a^2b^n=0$, that is $a=a^2b^n$. Similarly $(a-b^na^2)^2=0$ implies $a=b^na^2$. Therefore R is strongly γ(n)-regular ring . ■

Corollary 2.4 :

Every reduced γ(n)-regular ring is strongly regular ring .

Definition 2.5:

A ring R is called strongly π(n)-regular ring (for some integer n) if for every element $a\in R$, there exists $b\in R$ such that $a^n = a^{2n}b$.

Proposition 2.6 :

Let R be strongly π(n)-regular ring . Then a^n is strongly γ(n)-regular element for every $a\in R$.

Proof :

Since R is a strongly π(n)-regular ring , then for every $a\in R$ there exists $b\in R$ such that $a^n=a^{2n}b$. Hence $a^n = a^{2n}b = a^na^nb = a^n(a^nb)b = a^{2n}b^2 = . . . = a^{2n}b^n$. Therefore a^n is strongly γ(n)-regular element . ■

Theorem 2.7 :

Let I be an ideal of a strongly γ(n)-regular ring R . Then R/I is also strongly γ(n)-regular ring .

Proof :

Let $a+I \in R/I$, then $a\in R$. Since R is a strongly γ(n)-regular ring , then there exists $b\in R$ such that $a=a^2b^n$. So , $a+I = a^2b^n+I = (a+I)^2(b+I)^n$. Therefore R/I is strongly γ(n)-regular ring . ■

Lemma 2.8 : [9]

Let a be a none zero element in a reduced ring R . Then l(a) = R(a) and $l(a) = l(a^2)$.

Theorem 2.9 :

Let R be a reduced ring and r/l(a) is γ(n)-regular ring for every $a\in R$. Then R is strongly γ(n)-regular ring .

Proof :

Let $a\in R$, then R/l(a) is γ(n)-regular ring . So there exists $b+l(a)\in R/l(a)$ such that $a+l(a)=(a+l(a))(b+l(a))^n(a+l(a))$. Hence $a+l(a)=(ab^na)+l(a)$. So $a-ab^na\in l(a)$ .that is $(a-ab^na)a=0$ and $(1-ab^n)a^2=0$. Hence $1-ab^n\in l(a^2) = r(a^2) = r(a) \Longrightarrow a(1-ab^n) = 0 \Longrightarrow a = a^2b^n$. Similarly $a=b^na^2$ .Therefore R is a strongly γ(n)-regular ring . ■

Note that R is also γ(n)-regular ring , that is since R is strongly γ(n)-regular ring .

3 : Semi-commutative rings :

In this section we deal with the relation between strongly γ(n)-regular rings and γ(n)-regular rings which they are semi-commutative rings .

Definition 3.1 : [12]

A ring R is said to be a semi-commutative ring if every idempotent element is central .

Hence every reduced ring is semi-commutative ring .[12]

Theorem 3.2 :

For any ring R , the following statements are equivalent :

(1) R is strongly γ(n)-regular ring .

(2) R is a semi-commutative γ(n)-regular ring .

Proof :

(1)$\Longrightarrow$(2) . From corollary 2.4 .

(2)$\Longrightarrow$(1) . Let $0 \neq a \in R$, then there exists $b \in R$ such that $a=ab^na$. Since ab^n is an idempotent element and R is semi-commutative ring , then ab^n is central . So $(ab^n)a=a(ab^n)$. That is $a=ab^na=a^2b^n$. Similarly , $a=b^na^2$ [Since b^na is an idempotent element] . Therefore R is a strongly γ(n)-regular ring . ■

Proposition 3.3:

Let R be a semi-commutative γ(n)-regular ring . Then R is a unit regular ring .

Proof :

Let $a \in R$, then there exists $b \in R$ such that $a=ab^na$. Hence ab^n and b^na are idempotent elements . So they are central elements .

Now $ab^n = a(b^na)b^n = a(b^n(b^na)) = (ab^n)(b^na) = b^n(ab^n)a = b^na$.

Let $u=a+ab^n-1$ and $v=ab^n+ab^{2n}-1$. Since $ab^n=b^na$ and $a=ab^na$,

we have :

$uv = (a+ab^n-1)(ab^n+ab^{2n}-1)$

$= a^2b^n + a^2b^{2n} - a + ab^nab^n + ab^nab^{2n} - ab^n - ab^n - ab^{2n} + 1$

$= a + ab^n - a + ab^n + ab^nb^n - ab^n - ab^n - ab^nb^n + 1 = 1$.

Similarly , $vu = 1$ and $ava = a(ab^n+ab^{2n}-1)a = a$. Therefore R is a unit regular ring . ■

Corollary 3.4 :

Every strongly γ(n)-regular ring is a unit regular ring .

Proof :

Let R be a strongly γ(n)-regular ring , then it is reduced γ(n)-regular ring [Corollary 2.4] . Since every reduced ring is semi-commutative ring , then by [proposition 3.3] R is a unit regular ring . ■

Definition 3.5 :

A ring R is called clean if each element can be expressed as the sum of a unit and idempotent .

Corollary 3.6 :

Let R be a strongly γ(n)-regular ring . Then a^n is a unit γ(n)-regular element for each $a \in R$.

Proof :

Since R is a strongly γ(n)-regular ring , then R is a semi-commutative γ(n)-regular ring . Hence a is a unit regular element [Corollary 3.4] . Since R is γ(n)-regular ring , then for $a \in R$ there exists $b \in R$ such that $a = ab^na$. So ab^n and b^na are central idempotent elements . That is $a = ab^na = a^2b^n$.

Let $u = ab^{2n} - ab^n + 1$ and $v = a - ab^n + 1$, then $uv = vu = 1$ and $u^n = ab^nb^n - ab^n + 1$. Hence $a^n = a^nu^na^n$. Therefore a^n is a unit γ(n)-regular element . ■

Corollary 3.7 :

Let R be a reduced γ(n)-regular ring . Then for each $a \in R$:

(1) a^n is a unit γ(n)-regular element .

(2) $a = eu$ for some idempotent element $e \in R$ and unit element $u \in R$.

(3) a is a clean element .

Proof :

(1) follows from [Corollary 3.6] .

(2) and (3) follows from [12] . ■

In the following ; N(R) is the set of the nilpotent elements of R .

Proposition 3.8 :

Let R be a semi-commutative γ(n)-regular ring . Then for each $a \in R$, $a^t = eu + x$ for some idempotent element e , unit element e and nilpotent element x .

Proof :

Since R is a semi-commutative γ(n)-regular ring , then R/N(R) is strongly regular [12] . Hence R/N(R) is reduced , that is , semi-commutative ring . So for each $a+N(R) \in R/N(R)$, $a^t+N(R)$ is a unit γ(n)-regular element [Corollary 3.6] . Hence there exists a unit u+N(R) such that $a^t+N(R) = (a^t+N(R))(u^t+N(R))^n(a^t+N(R))$. That is u^n is a unit element in R [12] . Now since $a^t u^n + N(R)$ is an idempotent in R/N(R) then $(a^t u^n + N(R))^2 = (a^t u^n a^t u^n) + N(R) = a^t u^n + N(R)$. Then there exists an idempotent element $e \in R$ such that $a^t u^n + N(R) = e + N(R)$ [6] . So , $a^t + N(R) = (a^t u^n + N(R))((u^n)^{-1} + N(R)) = e(u^n)^{-1} + N(R)$. Hence , $a^t - e(u^n)^{-1} \in N(R)$. Therefore $a^t = e(u^n)^{-1} + x$ for some $x \in N(R)$. ■

4 : Quasi-commutative rings :

The definition of quasi-commutative rings was introduced by Kandasamy [15] , that is a ring R with $ab = b^n a$ for every pair $a,b \in R$ and for some positive integer n depending on a and b .

We now turn to the case where n is a fixed integer greater than one .

Definition 4.1 :

A ring R is said to be q(n)-commutative , where n is an integer greater than one , if for every $1 \neq a \in R$ and $b \in R$, $ab=b^n a$.

Proposition 4.2 :

Let R be a quasi-commutative γ(n)-regular ring , then R is strongly regular ring .

Proof :

Let $a \in R$, then there exists $b \in R$ such that $a = ab^n a$. Since R is a quasi-commutative ring , then $ab^n = (b^n)^t a = b^{nt} a$ for some positive integer t . Hence $a = ab^n a = b^{nt} aa = b^{nt} a^2$. Also we have $b^n a = a^m b^n$ for some positive integer m . Then $a = ab^n a = aa^m b^n = a^2(a^{m-1} b^n)$.
Therefore R is strongly regular ring . ■

Theorem 4.3 :

Let R be a q(n)-commutative ring . Then the following statements are equivalent :

(1) R is γ(n)-regular ring .

(2) R is strongly regular ring .

Proof :

(1) $\Longrightarrow$ (2) . Since R is a q(n)-commutative ring , then R is quasi-commutative ring . Hence by [proposition 4.2], R is strongly regular ring.

(2) $\Longrightarrow$ (1) . Since R is strongly regular ring , then for every $a \in R$ there exists $b \in R$ such that $a = a^2 b$.

Now since R is q(n)-commutative ring , then $ab = b^n a$. Hence $a = aab = ab^n a$. Therefore R is γ(n)-regular ring . ■

Corollary 4.4 :

Let R be a q(n)-commutative reduced ring . Then the following statements are equivalent :

(1) R is γ(n)-regular ring .

(2) R is regular ring .

Proof :

(1) $\Longrightarrow$ (2) . The proof is straight forward check which we omit .

(2) $\Longrightarrow$ (1) . Since R is reduced and regular , then R is strongly regular ring . Therefore R is γ(n)-regular ring [Theorem 4.3] . ■

Corollary 4.5 :

The center of every reduced q(n)-commutative γ(n)-regular ring is γ(n)-regular ring .

Proof :

Let R be a reduced q(n)-commutative γ(n)-regular ring . Then R is regular ring . So cent R is regular [6] . Therefore cent R is γ(n)-regular ring [Corollary 4.4] . ■

Corollary 4.6 :

Let R be a q(n)-commutative reduced ring such that every ideal of R is a maximal right ideal , then R is γ(n)-regular ring .

Proof :

Since R is reduced and every prime ideal is maximal right ideal, then R is regular ring [7] .

Now since R is q(n)-commutative ring and reduced then R is γ(n)-regular ring [Corollary 4.4] . ■

Corollary 4.7 :

Let R be a q(n)-commutative reduced ring such that R/P is γ(n)-regular ring for every prime ideal P in R . Then R is γ(n)-regular ring .

Proof :

Let P be a prime ideal in R . Then R/P is division ring , because R/P is a γ(n)-regular ring and with nonzero divisors . Therefore P is a maximal right(left) ideal in R and R is γ(n)-regular ring [Corollary 4.6]. ■

Proposition 4.8 :

Let R be a q(n)-commutative ring . Then the following statements are equivalent :

(1) R is γ(n)-regular ring .

(2) r(a) is direct summand for every $a \in R$.

Proof :

(1) $\Longrightarrow$ (2) . From proposition 1.3 .

(2) $\Longrightarrow$ (1) . Let $a \in R$. Since r(a) is direct summand , then $R = aR + r(a)$. Since $1 \in R$, then $1 = ar + d$ for some $r \in R$ and $d \in r(a)$. So ,

$a = a^2r + ad$ .That is $a = a^2r$. Since R is q(n)-commutative ring , then $a = a(ar) = ar^na$. Therefore R is γ(n)-regular ring . ■

Theorem 4.9 :

Let R be a q(n)-commutative reduced ring . Then the following statements are equivalent :

(1) R is γ(n)-regular ring .

(2) Every element $a\in R$ can be written as $a = ue$ for some idempotent $e\in R$ and unit $u\in R$.

Proof :

(1) $\Longrightarrow$ (2) . From Corollary 3.7 .

(2)$\Longrightarrow$(1).Let $a = eu$, where e is idempotent and u is unit,then $e = au^{-1}$. Now $ea = au^{-1}a$, but $ea = eeu = e^2u = eu = a$. So $a = au^{-1}a$ which is regular element .

Therefore by [Corollary 4.4] R is γ(n)-regular ring . ■

Theorem 4.10 :

Let R be a q(n)-commutative ring . Then the following statements are equivalent :

(1) R is γ(n)-regular ring .

(2) For every right ideal I and left ideal J in R , $IJ = I \cap J$.

Proof :

(1) $\Longrightarrow$ (2) . Since R is γ(n)-regular ring , then it is regular . Hence (2) holds by [9;Theorem 1.1.7] .

(2) $\Longrightarrow$ (1) . Let $a\in R$. Since $a\in aR\cap Ra = aRa$. So $a = aba$ for some $b\in R$. Since R is q(n)-commutative ring, then $a = (ab)a = b^na^2$ .Hence R is strongly γ(n)-regular ring . Therefore R is γ(n)-regular ring. ■

Theorem 4.11 :

Let R be a q(n)-commutative reduced ring such that every principal left ideal of R is left annihilator , then R is γ(n)-regular ring .

Proof :

Since R is reduced and every principal left ideal of R is a left annihilator , then R is strongly regular ring [9] . Since R is q(n)-commutative , then by [Theorem 4.3] R is γ(n)-regular ring . ■

Proposition 4.12 :

Let R be q(n)-commutative ring . Then the following statements are equivalent :

(1) R is γ(n)-regular ring .

(2) Every principal ideal is a direct summand .

Proof :

(1) $\Longrightarrow$ (2) . Clearly from [10] .

(2) $\Longrightarrow$ (1) . Let $R = aR \oplus K$ for some ideal K of R . Since $1 \in R$, then $1 = ar + k$ for some $r \in R$ and $k \in K$.

Since R is q(n)-commutative ring , then $ar = r^n a$.

So , $1 = r^n a + k$. That is $a = ar^n a + ak$.

Hence $ak = a - ar^n a \in aR \cap K = \{0\}$. So $a = ar^n a$. Therefore R is γ(n)-regular ring . ■

Theorem 4.13 :

Let R be a q(n)-commutative reduced ring such that every maximal ideal of it is a right annihilator , then R is γ(n)-regular ring .

Proof :

Let $a \in R$. Suppose that $aR + r(a) \neq R$, then there exists a maximal right ideal M containing $aR + r(a)$.

So $M = r(b)$ for some $b \in R$.

Hence $b \in l(aR + r(a)) \subseteq l(a) = r(a)$ [9 ;Theorem 1.3.10] .

So $b \in M = r(b)$. Then $b^2 = 0$. Since R reduced , then $b = 0$, a contradiction . Hence $aR + r(a) = R$. Now $1 \in R$ implies $1 = ax + y$ for some $x \in R$ and $y \in r(a)$. So $a = a^2x + ay$, that is $a = a^2x$. Hence R is strongly regular ring .

Therefore R is γ(n)-regular ring [Theorem 4.3] . ■

Theorem 4.14 :

Let R be a q(n)-commutative ring . Then R is γ(n)-regular ring if and only if each ideal in R is a radical ideal .

Proof :

Suppose that R is γ(n)-regular ring . Then R is regular ring . Hence every ideal is radical ideal [11] .

Conversely : Let $I = \sqrt{I}$ for each ideal I in R . Take $I = a^2R = \sqrt{a^2R}$. Then $a^2 \in a^2R$. So $a \in \sqrt{a^2R} = a^2R$. That is $a = a^2b$ for some $b \in R$. Hence R is strongly regular ring . Therefore by [Theorem 4.3] R is γ(n)-regular ring . ■

Corollary 4.15 :

Let R be a q(n)-commutative ring . Then R is γ(n)-regular ring if and only if each ideal in R is semi-prime .

Theorem 4.16 :

Let R be a q(n)-commutative regular ring , then R is strongly γ(n)-regular ring .

Proof :

Let $a \in R$, then there exists $b \in R$ such that $a = aba$. Since R is q(n)-commutative ring , then $ab = b^n a$. So $a = a^2 b^n$. Therefore R is strongly γ(n)-regular ring . ■

Theorem 4.17 :

Let R be q(n)-commutative ring , then the following statements are equivalent :

(1) R is strongly γ(n)-regular ring .

(2) R is strongly regular ring .

Proof :

(1) $\Longrightarrow$ (2) . The proof is straight forward check which we omit .

(2) $\Longrightarrow$ (1) . Since R is strongly regular ring , then there exists $b \in R$ such that $a = a^2 b$. Since R is q(n)-commutative ring then $ab = b^n a$. Hence $a = ab^n a$. Since R is strongly regular ring , then R is reduced . So $a = a^2 b^n$. Therefore R is strongly γ(n)-regular ring . ■

Reference

[1] Arens R.F. and Kaplansky I. , Topological Representation of Algebras , Trans Amer. Math. Soc. 63 (1948), 457 – 481 .

[2] Azumaya G. , Strongly π-Regular Rings , J. Fac. Sci. Hokkaido Univ. Vol. 13(1954), 34 – 39.

[3] CHIN. A. Y. M. , Clean elements in abelian Proc. Indian Acad. Sci. (Math. Sci.) Vol. 119 , No.2 , (2009) , pp. 145 – 148.

[4] Drazen M.P. , Rings with Central Idempotent or Nilpotent elements , Proc. Edinburgh Math. Soc. 9(1958), 157 – 165.

[5] Ehrlich G. , Unit Regular Rings , Portugal. Math. Vol. 27(1968), 209 – 212.

[6] Goodearl , K.R. " **Von Neumann Regular Rings** " (1979), Pitman , London.

[7] Lam T. Y. , **A First Course in Noncommutative Rings** , Springer Verlag , New York , Inc. , (1991).

[8] Luth J. , A note on strongly regular rings , Proc. Japan Acad. , Vol. 40(1964) , 74 – 75 .

[9] Mahmood A. S. , On Von Neumann Regular Rings , M. Sc. Thesis , Mosul University (1990).

[10] Mohammad A. J. and Salih S. M. , On γ-regular rings , J. Edu. Sci. Vol. (18) No. 4(2006) , 118 – 139 .

[11] Naoum A. G. and Kider J. R. , Commutative π-Regular Rings , Iraq J. Sci. , Vol. 35 , No. 4(1994), 1169 – 1184 .

[12] Naoum F. S. , On π-Regular Rings , Ph.D. Thesis , Mosul University , (2004) .

[13] Neumann J. V. ,Regular Rings , Proc. Nat. Acad. Sci. U. S. A. Vol. 22(1936) , 707 – 713 .

[14] Osba , E. A. , Henriksen , M. , Alkam , O. and Smith , F. H. " The maximal regular ideal of some commutative rings " , Comment. Math. Univ. Carolina , Vol. 47(2006) , pp. 1 – 10 .

[15] Vasantha Kandasamy W. B. , On Quasi-Commutative Rings , Aribb. J. Math. Comput. Sci. Vol. 5 , No. 1&2(1995) , 22 – 24 .