College of Basic Education Research Journal

www.berj.mosuljournals.com

The moment for nonlinear (exponential) stochastic differential equations

Abdulghafoor Jasim Salim
Department of Mathematics, college of computer science and Mathematics, University of Mosul.
Jalal Khalaf Qassim
Department of Mathematics, college of computer science and Mathematics, University of Mosul.

Article Information

Article history:

Received: September 10.2023

Reviewer: January 9.2024

Accepted: : January 18.2024

Key words :
exponential stochastic differential equations; Itô's formula; stochastic integral; moments.

Correspondence:

Abstract

in this article researcher focused on moments for exponential stochastic differential equations by using Ito's-formula. First he found general form for exponential stochastic differential equation. then, derived their moments (mean, variance, and k.th moments). Researcher also provided some examples to explain the method. Stochastic differential equations (SDEs) are frequently employed in various modeling applications due to their ability to incorporate randomness or uncertainty into ordinary differential equations. By introducing a random or stochastic component, these equations can capture unexpected phenomena. Consequently, SDEs are also known as stochastic or random differential equations, with the noise term representing the random element. In this way, an SDE comprises multiple random processes, leading to the solution itself being a stochastic process. Consider the ordinary differential equation represent the differential form of Brownian motion's, multiply then the exact solution will be needed in order to find the moments to the solution of the exponential stochastic differential equations.

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ISSN: 1992 – 7452

- **Introduction:**

Stochastic differential equations (SDEs) are frequently employed in various modeling applications due to their ability to incorporate randomness or uncertainty into ordinary differential equations. By introducing a random or stochastic component, these equations can capture unexpected phenomena. Consequently, SDEs are also known as stochastic or random differential equations, with the noise term representing the random element. In this way, an SDE comprises multiple random processes, leading to the solution itself being a stochastic process. Consider the ordinary differential equation

$$\begin{cases} x(t) = f\big(t, x(t)\big) \;; \quad t > 0 \\ x(0) = x_0 \end{cases} \qquad \dots(1)$$

$where\ f(0)\ is\ a\ smooth\ fuction\ and\ x_0$ is any fixed point $x_0 \in \mathcal{R}^n$

In the S.D.E. model such a model must contain random or stochastic effects called (wiener process known as Brownian motion) in order to formulate random behavior, then we obtain

$$\{\dot{x}(t) = f\big(t, x(t)\big) + g(t, x(t))\zeta(t) \quad ; \quad t > 0\,. \qquad \dots(2)$$

Where the process $\{\zeta(t)\}$ is a white noise (defined as the formal derivative of Weiner process).

Then we can write (2) as:

$$\Big\{\frac{dx(t)}{dt} = f(t, x(t)) + g(t, x(t)\frac{dw(t)}{dt} \qquad \dots(3)$$

Where$\frac{dw(t)}{dt} = \xi(t)$, $dw(t)$represent the differential form of Brownian motion's, multiply by dt so equation (3) became

$$\begin{cases} dx(t) = f\big(t, x(t)\big)dt + g(t, x(t))dw(t) \\ x(0) = x_0 \end{cases} \qquad \dots(4)$$

Where $f(0)$is the drift coefficient and $g(0)$is the diffusion coefficient

Equation (4) is called (S.D.E) stochastic differential equation

After we find the general form of the exponential stochastic differential equation by applying Ito formula, we also find the exact solution for those equation in order to find the moment for the exact solution for the proposed model

PREREQUISITES AND RESULTS:

In this paragraph, we give some basic definitions which we have needed and find the general form of the exponential stochastic differential equation.

Definition(1):**The random variable[3]**

The variable at random $(\mathfrak{r}, v)$ is a function that translates from the sample space to the real number $\mathcal{R}$ (i.e., $\mathcal{R}$)

Definition (2): stochastic process [4]

A stochastic process, also known as a random process, is a mathematical entity that consists of a collection of random variables represented by x(t),t belonging to $\mathcal{T}$ (sorted by index set $\mathcal{T}$ where $\mathcal{T} \in \mathcal{R}$)

Definition (3): (**Wiener** (Brownian motion) **Process) [5]**

The continuous-time stochastic wiener process (Brownian motion) $\{W(t)\}$over the interval $[0, \mathcal{T}]$, satisfying the following conditions:.

1: $w\ (0)\ =\ 0$

2: $If\ t, s\ \geq\ 0, then\ w(t) -\ w(s)$ is normally distributed with zero mean and variance $|t - s|$.

3: For $0 \leq s < t < k < j \leq \mathcal{T}, w$ (t)- w (s) and $w(j) - w(k)$ are independent increments.

Definition (4): Stochastic integral: [6]

The Stochastic integral is an integral defined as a sum of more than an integration and multiplied by the increase in time on the Brownian motion trajectory (Dumas and Luciano, 2017). Which is defined by

$$\int_a^b g(t) dw(t) = \sum_{i=0}^{\kappa-1} \zeta_i \left(w_{t_{i+1}} - w_{t_i}\right) \qquad \ldots(5)$$

Where $g(\mathrm{t}) = \{g(t)\}_{\mathrm{a} \leq \mathrm{t} < b}$ is a real valued stochastic process and $\{w(\mathrm{t})\}$ is a Wiener process(Brownian motion).

Let equation (4) be expressed in the form, that is,

$$\mathcal{X}(t) = \mathcal{X}(0) + \int_0^t f(\mathrm{s}, x(\mathrm{s}))\mathrm{ds} + \int_0^t g(\mathrm{s}, x(\mathrm{s}))\mathrm{d}w(\mathrm{s}) \quad \ldots(6)$$

Where the two integrals $\int_0^t f(\mathrm{s}, x(\mathrm{s}))\mathrm{ds}$, $\int_0^t g(\mathrm{s}, x(\mathrm{s}))\mathrm{d}w(\mathrm{s})$

are well-defined in order that equation (7) hold, and we have the following conditions.

$\mathrm{E}\int_0^t g^2(\mathrm{s}, x(\mathrm{s}))\mathrm{ds} < \infty, \int_0^t |f(\mathrm{s}, \mathrm{x}(\mathrm{s}))|\mathrm{ds} < \infty$ for all, almost surely $t \geq 0$

The stochastic integral has one of the most essential properties.

$$\int_0^t w(\mathrm{S})\mathrm{d}w(\mathrm{s}) = \frac{1}{2}\int_0^t d(w^2(S)) - \frac{1}{2}\int_0^t ds = \frac{1}{2}w^2(t) - \frac{t}{2} \quad \ldots(7)$$

Then the general form is that :

$$\int_0^t w^j(t)dw(t) = \frac{1}{j+1}w^{j+1}(t) + \sum_{\kappa=0}^{j-2}(-1)^{\kappa+1}\frac{j!}{2^{\kappa+1}(j-k)!}w^{j-\kappa}(t) + (-1)^j\frac{j!}{2^j}\frac{t}{2} \quad \ldots(8)$$

Definition (5): Ito's integral formula [7]:

Consider the stochastic differential equation, which has the following form.

$$d\mathrm{x}(t) = f(t, \mathrm{x}(t))dt + g(t, \mathrm{x}(t))dw(t) \quad \ldots(9)$$

If $F(\mathrm{t}, x(\mathrm{t}))$ is a smooth function for $0 \leq t \leq \mathcal{T}$, we have by the basic Taylor rule

$$\mathrm{dF}(\mathrm{t}, x(\mathrm{t})) = \frac{\partial \mathrm{F}}{\partial \mathrm{t}}\mathrm{dt} + \frac{\partial \mathrm{F}}{\partial \mathrm{x}}d\mathrm{x}(t) + \frac{1}{2}\frac{\partial^2 \mathrm{F}}{\partial \mathrm{x}^2}\mathrm{g}^2\mathrm{dt}$$

Then if the requisite partial derivatives are available, and the mixed differentials are merged according to the rules,

$$dt.dw = dw.dt = (dt)^2 = 0 \quad \& \ (dw)^2 = dt$$

So,

$$dF\big(t,x(t)\big)=\left(\frac{\partial F}{\partial t}+f\big(t,x(t)\big)\frac{\partial F}{\partial x}+\frac{1}{2}g^2\big(t,x(t)\big)\frac{\partial^2 F}{\partial x^2}\right)dt+g(t,x(t))\,\frac{\partial F}{\partial x}dw(t)$$
…(10)

Equation.(11) is called Ito's formula, where $x(t)$ satisfies Equation, (10).

Definition (6) : Expectation and Variance [8]

If x is a random variable specified on the probability space$(\Omega.\mathcal{F}.\mathcal{P})$, then x's expected values or mean are.

$E(x)=\ \mu=\sum_i x_i p(x_i).$ …(11)

Where $p(x_i)$ is the probability mass function

That's the average of x across all of the probability spaces.

While For a continuous random variable over $\mathcal{R}$.

$E(x)=\int_{-\infty}^{\infty} xf(x)dx$ …(12)

Provided that the integral exist ,here f(x) is the probability density function.

The variance is a measure of how widely data is distributed around the mean.

$Var(x)=E((x-\mu)^2)=E(x^2)-(E(x))^2$ …(13)

Definition (7): The kth-order moment:[9]

A continuous random variable's kth-order moment is defined as:

$E(x^\kappa)=\int_{-\infty}^{\infty} x^\kappa f(x)dx;$; f(x) is the probability density function.

And for discreet random variable

$E(x^\kappa)=\sum_i x_i^\kappa p(x_i)$; $p(x_i)$is the mass function.

Note:

From the definition of the Brownian motion process $\{w(t)\}$, since the increments

$w(t) - w(s)$ are normally distributed with mean zero and variance $|t - s|$,

$w(t) - w(s) \sim N(0,1) \sim |t - s|$, then

$$E[w(t)] = 0 \text{ and } Var[w(t)] = t \qquad \dots(14)$$

Their fore for any integer $\kappa \geq 0$ we have

$$E[w^{2\kappa}] = \frac{(2\kappa)!}{2^k \kappa!} t^\kappa, E[w^{2\kappa+1}] = 0 \qquad \dots(15)$$

In particular, $E[w^4] = 3t^2$, $E[w^6] = 15t^3$

$$E\left[\int_a^b f(w(t), t) dw\right] = 0 \qquad \dots (16)$$

main results:

In this paragraph we find the solution for general form of the exponential stochastic differential equation in order to find the moments by the use of Ito-integral formula

i) Let $F\big(t, \mathcal{X}(t)\big) = e^{(tx)^n}$ and suppose that x(t) satisfies equation(9) :

Let F(t,x(t)) and its partial derivative are continuous and smooth function for $0 \leq t \leq \mathcal{T}$, then

$$\frac{\partial F}{\partial t} = nt^{n-1} x^n e^{(tx)^n} \ , \frac{\partial F}{\partial x} = nx^{n-1} t^n e^{(tx)^n}$$

$$\frac{\partial^2 F}{\partial x^2} = n^2 t^{2n} x^{2n-2} e^{(tx)^n} + n(n-1) t^n x^{n-2} e^{(tx)^n}$$

By using Ito-integral formula

$$d(F) = \left[\frac{\partial F}{\partial t} + f \frac{\partial F}{\partial x} + \frac{1}{2} g^2 \frac{\partial^2 F}{\partial x^2} \right] dt + g \frac{\partial F}{\partial x} dw(t) \qquad \dots (17)$$

$$d\left(e^{(tx)^n}\right) = \Big[nt^{n-1} x^n e^{(tx)^n} + fnx^{n-1} t^n e^{(tx)^n} + \frac{1}{2} g^2 \left(n^2 t^{2n} x^{2n-2} e^{(tx)^n} + n(n-1) t^n x^{n-2} e^{(tx)^n}\right)\Big] dt + \left[gnx^{n-1} t^n e^{(tx)^n}\right] dw(t)$$

By taking the integral from 0 to t, the solution is:

$$\int_0^t d\left(e^{(tx)^n}\right) = \int_0^t [\Big[nt^{n-1}x^n e^{(tx)^n} + fnx^{n-1}t^n e^{(tx)^n} + \frac{1}{2}g^2\left(n^2t^{2n}x^{2n-2}e^{(tx)^n} + n(n-1)t^n x^{n-2}e^{(tx)^n}\right)\Big]dt + \int_0^t [\left[gnx^{n-1}t^n e^{(tx)^n}\right]dw(t)$$

$$\left(e^{(tx)^n}\right) = e^0 + \int_0^t [\Big[nt^{n-1}x^n e^{(tx)^n} + fnx^{n-1}t^n e^{(tx)^n} + \frac{1}{2}g^2\left(n^2t^{2n}x^{2n-2}e^{(tx)^n} + n(n-1)t^n x^{n-2}e^{(tx)^n}\right)\Big]dt + \int_0^t [\left[gnx^{n-1}t^n e^{(tx)^n}\right]dw(t)$$

Taking expectations to both sides

$$E\left(e^{(tx)^n}\right) = 1 + E\left(\int_0^t [\Big[nt^{n-1}x^n e^{(tx)^n} + fnx^{n-1}t^n e^{(tx)^n} + \frac{1}{2}g^2\left(n^2t^{2n}x^{2n-2}e^{(tx)^n} + n(n-1)t^n x^{n-2}e^{(tx)^n}\right)\Big]dt\right) + E\left(\int_0^t [\left[gnx^{n-1}t^n e^{(tx)^n}\right]dw(t)\right)$$

Since $\{w(t)\}$ is a wiener process and from the previous note

$$E\left(\int_0^t [\left[gnx^{n-1}t^n e^{(tx)^n}\right]dw(t)\right) = 0$$

Then

$$E\left(e^{(tx)^n}\right) = 1 + E\left(\int_0^t [\left[nt^{n-1}x^n e^{(tx)^n} + fnx^{n-1}t^n e^{(tx)^n} + \frac{1}{2}g^2\left(n^2t^{2n}x^{2n-2}e^{(tx)^n} + n(n-1)t^n x^{n-2}e^{(tx)^n}\right)\right] dt\right) \quad \dots (18)$$

To find the variance , we need to find $E[\left(e^{(tx)^n}\right)]^2$, that is $F(t,x(t))= (e^{t^n x^n})^2$

$$d(F(t,x(t))) = d(e^{t^n x^n})^2 = \left[2nt^{n-1}x^n e^{2t^n x^n} + 2fnx^{n-1}t^n e^{2t^n x^n} + \frac{1}{2}g^2\left(4n^2t^{2n}x^{2n-2}e^{2t^n x^n} + 2n(n-1)t^n x^{n-2}e^{2t^n x^n}\right)\right] dt + \left[2gnx^{n-1}t^n e^{2t^n x^n}\right]dw(t)$$

Taking the integration for both side from 0 to t , we get

$$\int_0^t d\,(e^{t^n x^n})^2 = \int_0^t [\left[2nt^{n-1}x^n e^{2t^n x^n} + 2fnx^{n-1}t^n e^{2t^n x^n} + 2g^2n^2t^{2n}x^{2n-2}e^{2t^n x^n} + g^2 n(n-1)t^n x^{n-2}e^{2t^n x^n}\right]dt + \int_0^t [\,2gnx^{n-1}t^n e^{2t^n x^n}(e^{t^n x^n})^2]dw(t)$$

$$(e^{t^n x^n})^2 = 1 + \int_0^t [\left[2nt^{n-1}x^n e^{2t^n x^n} + 2fnx^{n-1}t^n e^{2t^n x^n} + 2g^2n^2t^{2n}x^{2n-2}e^{2t^n x^n} + g^2 n(n-1)t^n x^{n-2}e^{2t^n x^n}\right]dt + \int_0^t [\,2gnx^{n-1}t^n e^{2t^n x^n}(e^{t^n x^n})^2]dw(t)$$

The expectation is then :

$$\begin{aligned}E(e^{t^n x^n})^2 = 1 &+ E\left(\int_0^t [\left[2nt^{n-1}x^n e^{2t^n x^n} + 2fnx^{n-1}t^n e^{2t^n x^n}\right.\right. \\ &+ 2g^2 n^2 t^{2n} x^{2n-2} e^{2t^n x^n} \\ &\left.\left. + g^2 n(n-1)t^n x^{n-2} e^{2t^n x^n}\right]dt\right) \qquad ....(19)\end{aligned}$$

From the definition of the variance , i.e.

$$Var\left(e^{t^n x^n}\right) = E(e^{t^n x^n})^2 - (E(e^{t^n x^n}))^2$$

$$\begin{aligned}Var\left(e^{t^n x^n}\right) = 1 &+ E\left(\int_0^t [\left[2nt^{n-1}x^n e^{2t^n x^n} + 2fnx^{n-1}t^n e^{2t^n x^n}\right.\right. \\ &\left.\left. + 2g^2 n^2 t^{2n} x^{2n-2} e^{2t^n x^n} + g^2 n(n-1)t^n x^{n-2} e^{2t^n x^n}\right]dt\right) - (1 \\ &+ E(\int_0^t [\left[nt^{n-1}x^n e^{(tx)^n} + fnx^{n-1}t^n e^{(tx)^n}\right. \\ &\left. + \tfrac{1}{2}g^2\left(n^2 t^{2n} x^{2n-2} e^{(tx)^n} + n(n-1)t^n x^{n-2} e^{(tx)^n}\right)\right] dt)^2\end{aligned}$$

The kth-order moment has the form:

$$\begin{aligned}d(e^{t^n x^n})^m = &\left[mnt^{n-1}x^n e^{mt^n x^n} + mfnx^{n-1}t^n e^{mt^n x^n}\right. \\ &\left. + \tfrac{1}{2}g^2\left(m^2 n^2 t^{2n} x^{2n-2} e^{mt^n x^n} + mn(n-1)t^n x^{n-2} e^{mt^n x^n}\right)\right] dt \\ &+ \left[mgnx^{n-1}t^n e^{mt^n x^n}\right]dw(t)\end{aligned}$$

By integration, we get

$$\int_0^t d\,(e^{t^n x^n})^m$$
$$= \int_0^t [mnt^{n-1}x^n e^{mt^n x^n} + mfnx^{n-1}t^n e^{mt^n x^n} + \tfrac{1}{2}g^2\big(m^2n^2t^{2n}x^{2n-2}e^{mt^n x^n} + mn(n-1)t^n x^{n-2}e^{mt^n x^n}\big)]dt + \int_0^t \big[mgnx^{n-1}t^n e^{mt^n x^n}\big]dw(t)$$

$$(e^{t^n x_t{}^n})^m = 1 + \int_0^t [mnt^{n-1}x^n e^{mt^n x^n} + mfnx^{n-1}t^n e^{mt^n x^n} + \tfrac{1}{2}g^2\big(m^2n^2t^{2n}x^{2n-2}e^{mt^n x^n} + mn(n-1)t^n x^{n-2}e^{mt^n x^n}\big)]dt + \int_0^t \big[mgnx^{n-1}t^n e^{mt^n x^n}\big]dw(t)$$

By taking the expectation, then we get

$$E(e^{t^n x_t{}^n})^m = 1 + E(\int_0^t \Big[mnt^{n-1}x^n e^{mt^n x^n} + mfnx^{n-1}t^n e^{mt^n x^n} + \tfrac{1}{2}g^2\big(m^2n^2t^{2n}x^{2n-2}e^{mt^n x^n} + mn(n-1)t^n x^{n-2}e^{mt^n x^n}\big)\Big])dt \qquad \dots(20)$$

ii) Let $F\big(t, \mathcal{X}(t)\big) = e^{(tx)^n}$ and suppose that x(t) satisfies equation(10) :

$$dx(t) = f\big(t, x(t)\big)dt + g\big(t, x(t)\big)dw(t)$$

$$\frac{\partial F}{\partial t} = nxe^{ntx} \quad , \frac{\partial F}{\partial x} = nte^{ntx} \quad , \frac{\partial^2 F}{\partial x^2} = n^2t^2e^{ntx}$$

By using Ito-integral formulas :

$$d(e^{ntx}) = \left[nxe^{ntx} + fnte^{ntx} + \tfrac{1}{2}g^2n^2t^2e^{ntx}\right]dt + [gn^2t^2e^{ntx}]dw(t)$$

$$\int_0^t d\,(e^{ntx}) = \int_0^t [nxe^{ntx} + fnte^{ntx} + \tfrac{1}{2}g^2n^2t^2e^{ntx}]\,dt + \int_0^t [gn^2t^2e^{ntx}]\,dw(t)$$

$$e^{ntx_t} = 1 + \int_0^t [nxe^{ntx} + fnte^{ntx} + \tfrac{1}{2}g^2n^2t^2e^{ntx}]\,dt + \int_0^t [gn^2t^2e^{ntx}]\,dw(t)$$

$$E(e^{ntx_t}) = 1 + E\left(\int_0^t \left[nxe^{ntx} + fnte^{ntx} + \tfrac{1}{2}g^2n^2t^2e^{ntx}\right]dt\right) \qquad \ldots(21)$$

To find the variance , we find $E(e^{ntx})^2$

$$d(e^{ntx})^2 = [2nxe^{2ntx} + 2fnte^{2ntx} + 2g^2n^2t^2e^{ntx}]dt + [2gn^2t^2e^{2ntx}]dw(t)$$

$$\int_0^t d\,(e^{ntx})^2 = \int_0^t [2nxe^{2ntx} + 2fnte^{2ntx} + 2g^2n^2t^2e^{ntx}]\,dt + \int_0^t [2gn^2t^2e^{2ntx}]dw(t)$$

$$(e^{ntx_t})^2 = 1 + \int_0^t [2nxe^{2ntx} + 2fnte^{2ntx} + 2g^2n^2t^2e^{ntx}]\,dt + \int_0^t [2gn^2t^2e^{2ntx}]dw(t)$$

$$E(e^{ntx_t})^2 = 1 + E\left(\int_0^t [2nxe^{2ntx} + 2fnte^{2ntx} + 2g^2n^2t^2e^{2ntx}]\,dt\right) \qquad \ldots(22)$$

From eq.(21) and eq.(22), we get

$$Var(e^{ntx}) = E(e^{ntx_t})^2 - (E(e^{ntx_t}))^2$$

$$Var(e^{ntx}) = 1 + E\left(\int_0^t [2nxe^{2ntx} + 2fnte^{2ntx} + 2g^2n^2t^2e^{ntx}]\, dt\right) - (1 + E\left(\int_0^t \left[nxe^{ntx} + fnte^{ntx} + \frac{1}{2}g^2n^2t^2e^{ntx}\right] dt\right))^2$$

The kth-order moment is:

$$d(e^{ntx})^m = \left[mnxe^{mntx} + mfnte^{mntx} + \frac{1}{2}m^2g^2n^2t^2e^{mntx}\right] dt + [mgn^2t^2e^{mntx}]dw(t)$$

$$\int_0^t d\,(e^{ntx})^m = \int_0^t \left[mnxe^{mntx} + mfnte^{mntx} + \frac{1}{2}m^2g^2n^2t^2e^{mntx}\right] dt + \int_0^t [mgn^2t^2e^{mntx}]\text{dw(t)}$$

$$(e^{ntx_t})^m = 1 + \int_0^t \left[mnxe^{mntx} + mfnte^{mntx} + \frac{1}{2}m^2g^2n^2t^2e^{mntx}\right] dt + \int_0^t [mgn^2t^2e^{mntx}]\text{dw(t)}$$

$$E(e^{ntx_t})^m = 1 + E(\int_0^t \left[mnxe^{mntx} + mfnte^{mntx} + \frac{1}{2}m^2g^2n^2t^2e^{mntx}\right] dt) \qquad \dots(23)$$

EXAMPLEs: To explain the method of finding the moment for exponential stochastic differential equation, first we find the solution by Ito-integral formula, then applied the previous method for some special cases:

EXAMPLE(1):

Let $dx(t) = dw(t)$; $x(0) = 0$, $\{w(t)\}$*is wiener process, that is* $w(0) = 0$

Solution: since $dx(t) = dw(t)$, then $x(t) = w(t)$ is a wiener process in the interval [0,t] and

Suppose $F = y_t = e^{ntx}$.

That is $\frac{\partial F}{\partial t} = nxe^{ntx}$, $\frac{\partial F}{\partial x} = nte^{ntx}$, $\frac{\partial^2 F}{\partial x^2} = n^2t^2e^{ntx}$

From equation (10)

$$d(F) = \left[\frac{\partial F}{\partial t} + f\frac{\partial F}{\partial x} + \frac{1}{2}g^2\frac{\partial^2 F}{\partial x^2}\right]dt + g\frac{\partial F}{\partial x}dw(t)$$

Or $d(y_t) = [\,nxe^{ntx} + \frac{1}{2}n^2t^2e^{ntx}]dt + nte^{ntx}dw(t)$ ...(24)

Taking expectation to both sides(25) of , we get

$$dE(y_t) = E((nxy_t + \frac{1}{2}\,n^2t^2y_t)dt) + E(nty_t dw(t)) \quad , \quad \text{since} \quad Ed(w_t)=0$$

$$dE(y_t) = (nE(x)E(y)_t + \frac{1}{2}\,E(n^2t^2y_t))dt$$

$$dE(y_t) = nE(x)E(y_t) + \frac{1}{2}(n^2t^2)E(y_t)dt$$

$$\frac{dE(y_t)}{E(y_t)} = \left[nE(x) + \frac{1}{2}(n^2t^2)\right]dt$$

By integrating from 0 to t

$\int_0^t \frac{dE(y_t)}{E(y_t)} = \int_0^t [nE(x) + \frac{1}{2}(n^2t^2)]dt$, since $x(t) = w(t)$ that is x follows a wiener stochastic process with zero mean , then ($E(x) = 0$)

$$\int_0^t \frac{dE(y_t)}{E(y_t)} = \int_0^t [\frac{1}{2}n^2s^2]ds$$

$$\ln(\,E(y_t) = \frac{1}{6}\,n^2t^3$$

$$E(y_t) = e^{\frac{1}{6}n^2t^3} \quad \rightarrow E(e^{ntx}) = e^{\frac{1}{6}n^2t^3} \qquad ...(25)$$

To find the variance , we need $E(e^{ntx})^2$

Let $E(e^{ntx})^2 = F_t$

$\frac{\partial \mathrm{F}}{\partial \mathrm{t}} = 2nxe^{2ntx}$, $\frac{\partial \mathrm{F}}{\partial \mathrm{x}} = 2nte^{2ntx}$, $\frac{\partial^2 \mathrm{F}}{\partial \mathrm{x}^2} = 4n^2t^2e^{2ntx}$

$d(F_{\mathrm{t}}) = [2nxe^{2ntx} + 2n^2t^2e^{2ntx}]dt + 2nte^{2ntx}dw(t)$

Taking expectation to both sides of SDE for $\mathrm{F_t}$, we have

$E(dF_{\mathrm{t}}) = E([2nxe^{2ntx} + 2n^2t^2e^{2ntx}]dt) + E(2nte^{2ntx}dw(t))$

$dE[F_{\mathrm{t}}] = \left[2nE[x]E[F_{\mathrm{t}}] + 2E[n^2t^2E[F_{\mathrm{t}}]\right]dt + E(2nte^{2ntx}dw(t))$

$dE[F_{\mathrm{t}}] = (2nE[x] + 2E[n^2t^2])\, E[F_{\mathrm{t}}]dt$, since $E[x]$ is wiener ($E[x] = 0$)

$$\int_0^t \frac{\mathrm{dE[F_t]}}{\mathrm{E[F_t]}} = \int_0^t 2n^2s^2\, ds$$

$$\ln(\, E(\mathrm{F_t}) = \frac{2}{3}\, n^2t^3$$

$E(\mathrm{F_t}) = e^{\frac{2}{3}n^2t^3}$ $\rightarrow E((e^{ntx})^2 = e^{\frac{2}{3}n^2t^3}$ …(26)

From eq.(26) and eq.(27), we get

$Var(e^{ntx}) = E(e^{ntx_{\mathrm{t}}})^2 - (E(e^{ntx_{\mathrm{t}}}))^2$

$Var(e^{ntx}) = e^{\frac{2}{3}n^2t^3} - (e^{\frac{1}{6}n^2t^3})^2$

$Var(e^{ntx}) = e^{\frac{2}{3}n^2t^3} - e^{\frac{1}{3}n^2t^3}$

The kth-moments , $E(e^{ntx})^m$

Let $E(e^{ntx})^m = F_{\mathrm{t}}$

$\frac{\partial \mathrm{F}}{\partial \mathrm{t}} = mnxe^{mntx}$, $\frac{\partial \mathrm{F}}{\partial \mathrm{x}} = mnte^{mntx}$, $\frac{\partial^2 \mathrm{F}}{\partial \mathrm{x}^2} = m^2n^2t^2e^{2ntx}$

$d(F_{\mathrm{t}}) = [mnxe^{mntx} + mn^2t^2e^{mntx}]dt + mnte^{2ntx}dw(t)$

Taking expectation on both sides of SDE for F_{t} , we have

$E(dF_{\mathrm{t}}) = E([mnxe^{mntx} + \frac{1}{2}m^2n^2t^2e^{mntx}]dt) + E(mnte^{mntx}dw(t))$

$dE[F_t] = [mnE[x]E[F_t] + E[\frac{1}{2}m^2n^2t^2E[F_t]]dt + E(mnte^{mntx}dw(t))$

$dE[F_t] = (mnE[x] + E[\frac{1}{2}m^2n^2t^2])\, E[F_t]dt$;(since $(E[x] = 0)$)

$\int_0^t \frac{dE[F_t]}{E[F_t]} = \int_0^t \frac{1}{2}m^2n^2s^2\, ds$

$\ln(E(F_t) = \frac{1}{6}m^2n^2t^3$

$E(F_t) = e^{\frac{1}{6}m^2n^2t^3} \quad \rightarrow E((e^{ntx})^m = e^{\frac{1}{6}m^2n^2t^3}$ ... (27)

EXAMPLE(2): Let $dx(t) = dt + dw(t)$,then find the moments of the solution ,Where *W is wiener process and* $e^{x(0)} = 1$. Let $F(t, x_t) = e^x$

Solution: since $dx(t) = dt + dw(t)$, that is $f = g = 1$ in interval $[0, t]$

We have $\frac{\partial F}{\partial t} = 0$, $\frac{\partial F}{\partial x} = \frac{\partial^2 F}{\partial x^2} = e^x$

From equation(10)

$dF(t, x_t) = \left[\frac{\partial F}{\partial t} + f\frac{\partial F}{\partial x} + \frac{1}{2}g^2\frac{\partial^2 F}{\partial x^2}\right]dt + g\frac{\partial F}{\partial x}dw(t)$

$d(e^x) = [\, e^x + \frac{1}{2}e^x]dt + e^x dw(t)$ ...(28)

Taking expectation to both sides of (28)

$Ed(e^x) = E[e^x + \frac{1}{2}e^x]dt + E[e^x dw(t)]$, Since $\{Ed(w_t) = 0\}$

Then

$Ed(e^x) = E[\, e^x + \frac{1}{2}e^x]dt$

Or

$dE(e^x) = E[\, e^x][\frac{3}{2}]dt$

$\frac{dE(e^x)}{E[\, e^x]} = \frac{3}{2}dt$

integrated both sides

$$\int_0^t \frac{\mathrm{dE}(e^{x_s})}{\mathrm{E}[\,e^{x_s}]} = \int_0^t \frac{3}{2} ds$$

$$\mathrm{E}[\,e^{x_t}] = 1 + e^{\frac{3}{2}t} - 1 \;\rightarrow \mathrm{E}[\,e^{x_t}] = e^{\frac{3}{2}t}$$

$$\mathrm{d}(e^{x})^2 = [2e^{2x} + \tfrac{1}{2}(4e^{2x})]dt + 2e^{2x}dw(t)$$

integral in both sides

$$\int_0^t \mathrm{d}(e^{x})^2 = \int_0^t [2e^{2x} + 2e^{2x}]dt + \int_0^t 2e^{2x}dw(t)$$

Expectation in both sides

$$E(\int_0^t \mathrm{d}(e^{x})^2) = E(\int_0^t [4e^{2x}]dt) + E(\int_0^t 2e^{2x}dw(t))$$

$$\int_0^t \mathrm{dE}(e^{x})^2 = E(\int_0^t [4e^{2x}]dt)$$

$$\int_0^t \mathrm{dE}(e^{x})^2 = 4\int_0^t E[e^{2x}]dt$$

$$\frac{1}{2}\int_0^t 2\frac{dE[e^{x_s}]^2}{E[e^{2x_s}]}=4\int_0^t ds$$

$$\frac{1}{2}In(\mathrm{E}(e^{x_s})^2|_0^t) = 4s|_0^t$$

$$In(\mathrm{E}(e^{x_s})^2|_0^t) = 8s|_0^t$$

$$\mathrm{E}(e^{x_t})^2 = 1 + e^{8s|_0^t}$$

$$\mathrm{E}(e^{x_t})^2 = 1 + \; e^{8t} - 1 \rightarrow \mathrm{E}(e^{x_t})^2 = \; e^{8t}$$

$$Var(e^{x_t}) = E(e^{x_t})^2 - (E(e^{x_t}))^2$$

$$Var(e^{x_t}) = \;\; e^{8t} - (e^{\frac{3}{2}t})^2$$

$Var(e^{x_t}) = \ e^{8t} - \ e^{3t}$

EXAMPLE(3) Let $dx(t) = dt + dw(t)\ ; F(t, x_t) = e^t$

Solution: since $dx(t) = dt + dw(t)$, then we get $f = \ g = 1$ in interval $[0, t]$

Let $F(t, x_t) = e^t$ be smooth function and their partial derivate exist and continuous.

That is $\frac{\partial F}{\partial t} = e^t, \ \ \frac{\partial F}{\partial x} = \frac{\partial^2 F}{\partial x^2} = 0$

By using equation (10)

$$dF(t, x_t) = \left[\frac{\partial F}{\partial t} + f\frac{\partial F}{\partial x} + \frac{1}{2}g^2\frac{\partial^2 F}{\partial x^2}\right]dt + g\frac{\partial F}{\partial x}dw(t)$$

$$d(e^t) = e^t dt$$

integrate both sides

$$\int_0^t e^t = \int_0^t e^t dt$$

Expectation in both sides

$$E(\int_0^t e^t) = \mathrm{E}(\int_0^t e^t dt)$$

$$E(e^t) = 1 + e^t - 1 \ \rightarrow E(e^t) = e^t \quad \dots (29)$$

$$\mathrm{d}(e^t)^2 = 2e^{2t}dt$$

integral in both sides

$$\int_0^t \mathrm{d}(e^t)^2 = \int_0^t 2e^{2t}dt$$

Expectation in both sides

$$\mathrm{E}(\int_0^t \mathrm{d}(e^t)^2) = E(\int_0^t 2e^{2t}dt)$$

$$\int_0^t \mathrm{dE}(e^t)^2 = E(\int_0^t [2e^{2t}]dt)$$

$$\mathrm{E}(e^t)^2 = 1 + e^{2t}\text{-}1 \rightarrow \mathrm{E}(e^{x_t})^2 = e^{2t} \qquad \dots (30)$$

From eq.(30) and eq.(31), we get

$$Var(e^t) = E(e^t)^2 - (E(e^t))^2$$

$$Var(e^t) = e^{2t} - (e^t)^2$$

$$Var(e^t) = e^{2t} - e^{2t}$$

$$Var(e^t) = \ 0$$

then the exact solution will be needed in order to find the moments to the solution of the exponential stochastic differential equation then we explain it with some examples**.**

Conclusion.

In this paper, after we find and prove the general form of exponential stochastic differential equations by using Ito- formula and using their theorem's .(That is, Ito's formula is valid for exponential form of the functions u(x(t), t) of the variables t and x).

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