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Stability of the Stochastic Differential Equation Using the Stratonovich – Formula

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Abstract

In this research, we applied and found the stability of some linear and non-linear stochastic differential equations after we used Stratonovich – formula to find their solutions , we applied the quadratic Lyapunov function which satisfied the given stochastic differential equation in order to use the Lyapunov direct second method and Lyapunov theorems to find and explain if the trivial (zero) solution was stable (p-stable, mean square stable) long with stochastically asymptotically stability, then we explained the methods with some examples.

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استقراريه المعادلة التفاضلية التصادفية باستخدام صيغة ستراتونوفيتش

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الخلاصة

تم في هذا البحث تطبيق وإيجاد الاستقرارية لبعض المعادلات التفاضلية التصادفية الخطية وغير الخطية بعد استخدام صيغة ستراتونوفيتش التكاملية لايجاد حلولها. وتم استخدام دالة ليابونوف التربيعية التي تحقق المعادلة التفاضلية التصادفية المعطاة من اجل استخدام الطريقة الثانية المباشرة والنظريات الخاصة بالاستقرارية للعالم ليابونوف لإيجاد ودراسة الاستقرارية من الرتبة p وكذلك مربع معدل الاستقرارية والاستقرارية المحاذية في الحجم الكبير، وتم عرض بعض الامثلة لتوضيح الطريقة.

الكلمات المفتاحية: الاستقرارية، المعادلة التفاضلية التصادفية (العشوائية)، صيغة ستراتونوفيتش، دالة ليابونوف.

1. Introduction:

Studying and applied stochastic differential equations (SDE) is a nature field of research. Different types of SDEs (linear or non- linear) have been used to model different phenomena in various areas, such as non-stable stock prices in finance (Merton, 1976)[1], the dynamics of some biological systems (Jha and Langmead, 2012)[2], filtering such as Kalman filter in navigation control in 1892[3], The stability means insensitivity of the state of the system to small changes in the initial state or the parameters of the system. For a stable system, the trajectories which are close to each other at a specific instant should therefore remain close to each other at all subsequent instants Lawrence C. E [4],the scientist Lyapunov in (1992) introduced the new concept of stability in a dynamical system[5]. Since this time, the concept of stability has been studied widely in different senses, Mao (2008) investigated different types of stabilities for stochastic differential equation[6], Nane, Ni (2017) and Wu (2016) are studying and extending the stability for the moments of SDES[7], Elbaz (2021) studied the stability of turbulent (linear and non-linear) systems by Lyapunov method approach[8].

In this paper we use the Stratonovich -integral formula for linear and nonlinear stochastic differential equation after assuming the quadratic Lyapunov function be given in order to

explain and applied the stability (Lyapunov direct method) with their theorems and we explain the methods by introducing some examples.

Suppose $\{x(t)\}$ satisfies the solution of the following stochastic differential equation

$$dx(t) = N\big(x(t)\big)dt + M\big(x(t)\big)dW(t), t \geq 0 \qquad (1)$$

Where $N(x(t), t) \in$ R, $N(x(t), t) \in$ R is measurable functions , with X(0) = x0 and W(t) is the standard Brownian process.

The integrating form of eq. (1) which is their solution, is:

$$x(t) = x(0) + \int_0^t N(X(s), s)ds + \int_0^t M(X(s), s)\, dW(s) \qquad (2)$$

suppose that at any initial value $x_t(0) = x_0 \in R^n$, there correspond a unique global solution denoted by $X(t; t_0; x_0)$.

Then equation (1) has the (zero (trivial) solution or equilibrium position) $x_t(0) \equiv 0$ corresponding to the given initial value $x_t(0)=0$.

Definition (1): (stable in probability) [9]:

If for every pair of (ε, r) where $\varepsilon \in (0,1)$ and $r > 0$ there exists $\delta = \delta(\varepsilon, r, t_0) > 0$ such that

$$P\{|x(t; t_0, x_0)| < r \; for \; all \; t \geq t_0\} \geq 1 - \varepsilon \qquad (3)$$

whenever $|x_0| < \delta_0$, then The trivial solution of equation (1) is stochastically stable or stable in probability . Otherwise, it is said to be unstable stochastically.

Definition (2): (positive-definite function) [10]

Assume that K denote the family of all continuous non-decreasing functions μ where $\mu: R_+ \to R_+$ such that if r and h are positive numbers, $\mu(0) = 0$ and $\mu(r) > 0$, let $V(x,t)$ be continuous function define on $S_h \times [t_0, \infty]$ *where* $S_h = \{x \in R^n: |x| < h\}$, hence the function $V(x,t)$ is said to be positive-definite if $V(0,t) \equiv 0$ and, for some $\mu \in K$,

$V(x,t) \geq \mu(|x|)$

for all $(x,t) \in S_h \times [t_0, \infty]$.

Also it is said to be negative-definite if $-V$ is positive-definite.

Definition (3): (p-stable) [11]

The trivial solution of

$$dx(t) = N\big(x(t)\big)dt + M\big(x(t)\big)dW(t), t \geq 0$$

for some $p > 0$ is called p-stable if for each $\in > 0$ there exists $\delta > 0$ such that $E|x(t,\emptyset)|^p < \in, t \geq 0$ provided that $||\emptyset||_1^p < \delta$.

Definition (4): (asymptotically stable stochastically) [12]

If the trivial solution is stochastically stable and, moreover, for every $\varepsilon \in (0,1)$ there exists $\delta = \delta(\varepsilon, r, t_0) > 0$ such that

$$P\left\{\lim_{n\to\infty} x(t; t_0, x_0) = 0\right\} \geq 1 - \varepsilon$$

whenever $|x_0| < \delta_0$, then The trivial solution of equation (1) is asymptotically stable stochastically.

Also if it is stochastically stable and for all $x_0 \in R^d$

$$P\left\{\lim_{n\to\infty} x(t; t_0, x_0) = 0\right\} = 1$$

Then the trivial (zero) solution of the equation (1) is asymptotically stable stochastically in the large.

Theorem (1): [13], [14] (Lyapunov theorem for ODE)

(i) The trivial(zero) solution is said to be stable, if we find a positive-definite function $V(t, X_t) \in C^{1,1}(S_h \times [t_0, \infty]; R_+)$ such that

For all $(x, t) \in S_h \times [t_0, \infty]$.

$$\dot{V}(x, t) = V_t(t, X(t)) + V_x(t, X(t))f(t, X(t)) \leq 0 \qquad (4)$$

(ii) The trivial(zero) solution is called asymptotically stable, if there exists a positive-definite decrescent function $V(t, X_t) \in C^{1,1}(S_h \times [t_0, \infty],, R_+)$ such that the derivative of $V(t, X_t)$ is negative-definite.

Definition (5): (asymptotically mean square stable) [14]

The trivial solution of the following system

$$dx(t) = N(x)dt + M(x)dw(t) \qquad (5)$$

is said to be asymptotically mean square stable on the interval $[0,\infty)$ *if it is stable and moreover,*

$$\lim_{t\to\infty} E^{(1)} [\|X(t)\|^2] = 0 \qquad (6)$$

That is it satisfies the following limitations in the neighborhood of the point $0 \in R^m$:

$$\lim_{t\to\infty} E^{(2)} [X(t)] = \lim_{t\to\infty} E^{(1)} \{X(t)X^T(t)\} = 0 \qquad (7)$$

Theorem (2):(Lyapunov stability for SDE) [15]

i): If we have a positive-definite function $V(y,t) \in C^{2,1}(S_h \times [t_{0,}\infty), R_+)$ such that, $LV(y, t) \leq 0$ *for* all $(y,t) \in S_h$ $x(t_{0,} \infty)$, then the (zero) trivial solution equation(1) is stochastically stable.

ii): If there exists a decrescent function $V(y, t) \in C^{1,2}(S_h \times [t_{0,}\infty), R_+)$,then the trivial(zero) solution of the given equation is asymptotically stable stochastically if $LV(y, t)$ is negative-definite.

iii): If there exists a decrescent radially unbounded function $V(y,t) \in C^{1,2}(R^n \times [t_{0,}\infty), R_+)$,then the simple zero solution of the given equation (equation(1)) is asymptotically stable stochastically in the large if such that $LV(y,t)$ is negative-definite.

2. PREREQUISITES AND RESULTS:

Suppose we have the quadratic Lyapunov function $V(X_t)$ is given

$$V(X_t) = X_t^T Q X_t$$

Where Q is an $m \times m$ symmetric positive definite matrix.

Lyapunov stability for the solution of stochastic differential equation:
let $F(t, X(t))$ be a smooth function and set $F(t, X(t)) = V(t, X_t)$ and suppose that it satisfies the existence of solution of equation (1), then we can write it by using Stratonovich - formula as:

$$dV(t, x_t) = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial X} N(,t, x_t) + \frac{1}{2}\frac{\partial V}{\partial X}\frac{\partial M}{\partial X} + \frac{1}{2}\frac{\partial^2 V}{\partial X^2} M(t, x_t)^2\right) dt + \frac{\partial V}{\partial x} M(t, x_t) dW_t \quad \ldots (8)$$

or we can write it as:

$$dV(t, x_t) = LV(t, x_t)dt + \frac{\partial V}{\partial X} M(t, x_t) dW_t \qquad \ldots (9)$$

The function $LV(X_t) \leq 0$ for stochastic differential equation is equivalence with $\dot{V}^{\cdot}(X_t) \leq 0$ for deterministic (ordinary)equation.

1): Nonlinear stochastic differential equation:

Proposition1: suppose we have equation (9) , that is

$$dV(t, X_t) = LV(t, x_t)dt + \frac{\partial V}{\partial X} M(t, x_t) dW_t \quad \text{where} \quad V(X_t) = X_t^T Q X_t$$

then

$$LV(x_t) = X_t^T Q N(t, X) + N(t, x_t)^T Q X_t + M(t, x_t)^T Q M(t, x_t) + X_t^T Q M_X(t, X)$$

Proof: from eq. (8) we have

$LV(t, x_t) = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial X} N(,t, x_t) + \frac{1}{2}\frac{\partial V}{\partial X}\frac{\partial M}{\partial X} + \frac{1}{2}\frac{\partial^2 V}{\partial X^2} M(t, x_t)^2\right)$, and since , $\frac{\partial V}{\partial t} = V_t(t, X(t)) = 0$, $\frac{\partial V}{\partial X} = V_x(t, X(t)) = 2QX_t^T$ and $\frac{\partial^2 V}{\partial X^2} = V_{xx}(t, X(t)) = 2Q$

then

$$dV(t,x_t) = [2X_t^T Q N(t,X) + M(t,x_t)^T Q M(t,x_t) + X_t^T Q M_X(t,X)]dt + [2X_t^T Q\ ]M(t,x_t)]dW_t$$

That is:

$$LV(x_t) = [(2X_t^T Q N(t,X) + M(t,x_t)^T Q M(t,x_t) + X_t^T Q M_X(t,X)\ ]$$

… (10)

since Q is symmetric matrix and $N(t,X)$ is smooth function , we can write equation (10) as:

$$LV(x_t) = X_t^T Q N(t,X) + N(t,x_t)^T Q X_t + M(t,x_t)^T Q M(t,x_t) + X_t^T Q M_X(t,X)$$

.(11)

which is equivalence with

$$LV(t,x_t) = V_t(t,X(t)) + V_x(t,X(t))N(t,X) + \frac{1}{2} V_x(t,X(t))\, M_X(t,X) + \frac{1}{2} trace M(t,x_t)^T V_{xx}(t,X(t)) M(t,x_t)$$

Propsitio2: suppose we have the following equation

$$dx(t) = N\big(x(t)\big)dt + M\big(x(t)\big)\mathrm{o}dW(t)$$

Then it is Stochastically asymptotically stable in the large if :

$$P\left\{\lim_{n\to\infty} x(t;t_0,x_0) = 0\right\} = 1$$

or

$$\lim_{k\to\infty} E^2\{X_t\} = 0$$

Proof:

From the theorem we need to prove that $LV(x_t)$ is negative-definite in neighborhood of $x_t = 0$ for $t \geq t_0$.

Since $dv(X_t) = V(X_t + dx_t) - V(X_t) = (X_t + dX_t)^T Q(X_t + dX_t) - X_t^T Q X_t$

then

$$dx(t) = N\big(x(t)\big)dt + M\big(x(t)\big) \circ dW(t)$$

Where

$$M\big(x(t)\big)\circ dW(t) = M\big(x(t)\big)dW(t) + \frac{1}{2}M_X(\mathrm{t},\mathrm{X})dt$$

$$dx(t) = N\big(x(t)\big)dt + M\big(x(t)\big)dW(t) + \tfrac{1}{2}M_X(\mathrm{t},\mathrm{X})dt \qquad ...(12)$$

then

$$dV(X_t) = [X_t^T + N(t,x_t)^T dt + M(t,x_t)^T dw_t + \tfrac{1}{2}M_X(\mathrm{t},\mathrm{X})dt\,]Q[X_t + N(t,x_t)dt + M(t,x_t)dw_t + \tfrac{1}{2}M_X(\mathrm{t},\mathrm{X})dt] - X_t^T Q X_t$$

$$= X_t^T Q X_t + QN(t,X_t)dt + X_t^T QM\,(t,X_t)\,dw_t + X_t^T \tfrac{1}{2}M_X(\mathrm{t},\mathrm{X})dt +$$

$$N(t,x_t)^T dtQX_t + N(t,Xt)^T dtQN(t,x_t)dt + N(t,x_t)^T dtQM(t,x_t)dw_t + N(t,x_t)^T dt\,Q\tfrac{1}{2}M_X(\mathrm{t},\mathrm{X})dt +$$

$$M(t,x_t)^T dw_t Qx_t + M(t,x_t)^T dw_t QN(t,x_t)dt + M(t,x_t)^T dw_t QM(t,x_t)dw_t + M(t,x_t)^T dw_t Q\frac{1}{2}M_X(\mathrm{t},\mathrm{X})dt +$$

$$\frac{1}{2}M_X(t,X_t)dtQX_t + \frac{1}{2}M_X(t,X_t)dtQ\,N(t,x_t)dt + \frac{1}{2}M_X(t,X_t)dtQM(t,X_t)dw_t + \frac{1}{2}M_X(t,X_t)dtQ\frac{1}{2}M_X(t,X_t)dt$$

$$-x_t^T Q x_t$$

By use the rules $dt.dt = dt.dW_t = dw_t.dt = 0, dw_t.dw_t = dt$

Then We get:

$$dV(x_t) = x_t^T QN(t,x_t)dt + x_t^T QM(t,x_t)dw_t + X_t^T\frac{1}{2}M_X(\mathrm{t},\mathrm{X})dt + N(t,x_t)^T dtQx_t + M(t,x_t)^T dw_t Qx_t + M(t,x_t)^T QM(t,x_t)dt + \frac{1}{2}M_X(\mathrm{t},\mathrm{X})dtQX_t$$

By taking the expectation for both sides, and since $\{W_t\}$ is wiener process which have the property $E(W_t) = 0$, then we get

$$E\{dV(x_t)\} = x_t^T QN(t, x_t)dt + N(t, x_t)^T QX_t d_t + M(t, x_t)^T QM(t, x_t)dt \\ + \mathrm{X}_\mathrm{t}^\mathrm{T} QM_X(\mathrm{t}, \mathrm{X})dt = LV(x_t)dt \,.$$

$-LV(x_t) \geq KV(X_t)$; $K = const$.

$\frac{d}{dt}E\{V(X_t)\} \leq -KE\{V(X_t)\}$, or $\frac{dE\{V(X_t\}}{E\{V(X_t\}} \leq -Kdt$

Then $lnE\{V(X_t)\} \leq -Kt$

$E\{V(X_t)\} \leq \exp(-Kt)$.

and since

$$\lim_{k\to\infty} E^2\{X_t\} = \lim_{k\to\infty} \exp(-2Kt) = 0$$

Therefore equation (12) is asymptotically stable in large , and the trivial solution is unstable if $LV(x_t)$ is positive-definite in some neighborhood of X_t = 0.

2): linear stochastic system differential equation:

Suppose we have the following linear system stochastic differential equation

$$dx_t = \propto x_t dt + bx_t \mathrm{o} dw_t \quad t \geq 0 \qquad \ldots (13)$$

where ∝, b are m×m constant matrices, her $N(x(t), t) = \propto x_t \; and \; M(x(t), t) = bx_t$

we can write equation (1٣) as

$$dx_t = \propto x_t dt + bx_t dw_t + \frac{1}{2}\beta dt$$

Then

$$LV(x_t) = {x_t}^T \propto^T Qx_t + {x_t}^T Q \propto x_t + {x_t}^T b^T Qbx_t + x_t Qb \qquad \text{....(14)}$$

Proof: In the same method as for nonlinear stochastic differential equation, we compute the Lyapunov function:

$$dV(x_t) = V(x_t + dx_t) - V(x_t) = (x_t^T + dx_t^T)Q(x_t + dx_t) - x_t^T Qx_t$$
$$= (x_t^T + \left(\propto {x_t}^T dt + (\beta x_t)^T d\beta_t + \frac{1}{2}\beta dt\right) Q \left(x_t + (\propto x_t)dt + \beta x_t d\beta_t + \frac{1}{2}\beta dt\right) - x_t^T Qx_t$$

$$= x_t^T Qx_t + x_t^T Q \propto x_t dt + x_t^T \beta x_t d\beta_t + x_t^T Q \frac{1}{2}\beta dt + (\propto x_t)^T dtQx_t + (\propto x_t)^T dtQ(x_t dt + (\propto x_t)^T dtQ\beta x_t d\beta_t + \propto x_{t)}{}^T dtQ\frac{1}{2}\beta dt + (\beta x_t)^T d\beta_t Qx_t + (\beta x_t)^T d\beta_t Q \propto x_t dt + (\beta x_t)^T d\beta_t Q\beta x_t d\beta_t + (\beta x_t)^T d\beta_t Q\frac{1}{2}\beta dt + \frac{1}{2}\beta dtQx_t + \frac{1}{2}\beta dtQ(\propto x_t)dt + \frac{1}{2}\beta dtQ\beta x_t d\beta_t + \frac{1}{2}\beta dtQ\frac{1}{2}\beta dt - x_t^T Qx_t$$
$$= x_t^T Q \propto x_t dt + x_t^T Q\beta x_t d\beta + (\propto x_t)^T dtQx_t + (\propto x_t)^T dtQ(\propto x_t)dt$$
$$= E\{dV(x_t)\} = x_t^T Q \propto x_t dt + (\propto x_t)^T Qx_t dt + (\beta x_t)^T Q\beta x_t dt + x_t Q\beta = LV(x_t)dt$$

That is

$$LV(x_t) = {x_t}^T \propto^T Qx_t + {x_t}^T Q \propto x_t + {x_t}^T b^T Qbx_t + x_t Qb$$

For stability , we need $LV(x_t) \leq 0$, that is

$$[{x_t}^T \propto^T Qx_t + {x_t}^T Q \propto x_t + {x_t}^T b^T Qbx_t + x_t Qb] \leq 0 \qquad (15)$$

After we find the values that satisfies the above equation (15), this explain how to find the stability of the given equation.

For asymptotically stability we must have

$\lim_{k\to\infty} E^2\{X_t\}\ equal\ to\ zero$.

3. Examples: we give some examples in order to apply and explain the methods.

Ex(1) : let $\{X_t\}$ satisfies the solution of the following Non-linear stochastic differential equation

$$dX_t = \left(a^2 {X_t}^3\right)dt + aX_t^2 \circ d\mathcal{W}_t$$

Where a is constant and $\mathcal{W}_t$ *is the wiener process*. determine the Lyapunov function and the stability.

Solution: we can write the given equation as,

$$dX_t = \left(a^2 {X_t}^3\right)dt + aX_t^2 d\mathcal{W}_t + aX_t dt$$

or

$$dX_t = \left(a^2 {X_t}^3 + aX_t\right)dt + aX_t^2 d\mathcal{W}_t$$

$$N(t, X_t) = (a^2 {X_t}^3)\ ; M(t, X_t) = aX_t^2$$

we find the stability by using Lyapunov function with Stratonovich_formula ,

let $\mathrm{V}(t, \mathrm{X_t}) = \mathrm{V}(\mathrm{X_t}) = \mathrm{X_t^T Q X_t}$, then

$$dV(X_t) = V(X_t + dx_t) - V(X_t) = (X_t + dX_t\ )^T Q(X_t + dX_t) - X_t^T QX_t$$

$$= \left(X_t + \left(a^2 {X_t}^3 + aX_t\right)dt + aX_t^2 d\mathcal{W}_t\ \ \right)^T Q(X_t + \left(a^2 {X_t}^3 + aX_t\right)dt + aX_t^2 d\mathcal{W}_t) - X_t^T QX_t$$

$$dV(X_t) = x_t^T Qx_t + x_t^T Q\left(a^2 {X_t}^3 + aX_t\right)dt + x_t^T Q\ aX_t^2 dw_t + \left(a^2 {X_t}^3 + aX_t\right)dtQx_t + \left(a^2 {X_t}^3 + aX_t\right)dtQ\left(a^2 {X_t}^3 + aX_t\right)dt + \left(a^2 {X_t}^3 + aX_t\right)dtQ\ aX_t^2 d\mathcal{W}_t + aX_t^2 d\mathcal{W}_t Qx_t + \left(aX_t^2 d\mathcal{W}_t Q\left(a^2 {X_t}^3 + aX_t\right)dt + aX_t^2 d\mathcal{W}_t\ \mathrm{Q}\ aX_t^2 d\mathcal{W}_t - X_t^T QX_t\ = \left(a^2 {X_t}^4 + ax_t^2\right)dt + ax_t^3 dw_t + \left(a^2 {X_t}^4 + ax_t^2\right)dt + ax_t^3 dw + a^2 x_t^4 dt\right] = \left(a^2 {X_t}^4 + ax_t^2 + a^2 {X_t}^4 + ax_t^2 + a^2 x_t^4\right)dt + 2ax_t^2 dw = (3a^2 x^4 + 2ax_t^2)dt + 2ax_t^3 dw$$

Then $dE(V(\ X_t\ )) = (3a^2 x^4 + 2ax_t^2)dt = LV(t, x_t)dt$

To apply Theorem (2), we need to prove that there exists a neighborhood of the zero point , where the function $\mathrm{LV}(X_t) = 3a^2x^4 + 2ax_t^2 \leq 0$. This holds iff the following inequality is satisfied: $X_t \leq (\frac{-2}{3a})^\wedge \frac{1}{2}$

That is, $\mathrm{LV}(t, \mathrm{X_t})$ is the function has negative values at each point $\leq X_t \leq (\frac{-2}{3a})^\wedge \frac{1}{2}$, and since $\mathrm{LV}(0) = 0$. Therefore, we conclude that there exists a neighborhood in which the function $\mathrm{LV}(X_t) = 3a^2x^4 + 2ax_t^2$ is negative definite. So, the trivial(zero) solution of the considered equation is asymptotically mean square stable on the interval $[0, \infty)$ since $\lim_{k\to\infty} E^2\{X_t\}\ equal\ to\ zero$, i.e.

Ex: (2): suppose we have the following stochastic differential equation:

$$dX_t = 3X_t dt + \exp(t)^2 \circ dw_t \qquad .(15)$$

Find and explain the stability.

Solution:

let $\mathrm{V(X_t) = X_t^T Q X_t}$, since

$$dX_t = 3X_t dt + \exp(t)^2 dw_t$$

$$dV(X_t) = V(X_t + dx_t) - V(X_t) = (X_t + dX_t\ )^T Q (X_t + \mathrm{d}X_t) - X_t^T Q X_t$$

or,

$$dV(X_t) = X_t^T(3X_t)dt + X_t^T exp\ t^2 dw_t + (3X_t)^T X_t dt \\ + \exp(t^2)X_t dw + (\exp t^2)^T \exp(t^2)dt$$

By taking the expectation, we get

$$E\big(dV(X_t)\big) = 3X_t^2 dt + 3X_t^2 dt + exp2t^2 dt = (6X_t^2 + exp2t^2)dt$$

That is $E\big(dV(X_t)\big) = LV(X_t)dt$

Then the stability condition is $(6X_t^2 + exp2t^2) \leq 0$ which is hold if and only if $X_t \leq (\frac{-(e^{2t^2})}{6})^{1/2}$

And since

$\frac{(dEV(X_t))}{E(V(X_t))} = (6X_t^2 + exp2t^2)dt$

$$lnE(V(X_t)) = \int_0^t (6X_s^2 + exp2s^2)ds$$

$E(V(X_t)) = \exp\left(\int_0^t (6X_s^2 + exp2s^2)ds\right)$, then

$\lim_{k\to\infty} E^2\{X_t\} = \lim_{k\to\infty} \exp\left(2\left(\int_0^t (6X_s^2 + exp2s^2)ds\right)\right) \neq 0$, then the stochastic differential equation is not asymptotically stochastically stable.

Ex: (3): (linear model)

$$dX_t = 6X_t dt + 4X_t \circ dW_t \quad (16)$$

Explain the stability.

Solution:

$dX_t = 6X_t dt + 4X_t \circ dW_t$, that is

$dX_t = (6X_t + 2)dt + 4X_t\, dW_t$

From equation(14), $LV(x_t) = 21x_t^2 + 4x_t$. (where Q=1).

Let $\mathrm{V(X_t) = X_t^T Q X_t}$.

Then

$$dV(x_t) = V(x_t + dx_t) - V(x_t) = (x_t + dx_t)^T Q(x_t + dx_t) - x_t^T Q x_t$$
$$dV(x_t) = x_t^T Q(6X_t + 2)dt + x_t^T Q(4x)dw_t + (6X_t + 2)dtQx_t + 4xdw_tQx_t + (4x)^T Q(4x)dt$$
$$= 6x_t^2 dt + 2x_t\ dt + 6x_t^2 dt + 2x_t\ dt + 16x^2 dt + 8x^2 dw$$

$= 28x_t^2 dt + 4x_t \ dt + 6x^2 dw$
$\therefore LV(x_t) = 28x_t^2 + 4x_t \ \geq 0$
Then the stability condition is $(28x_t^2 + 4x_t \ ) \leq 0$ which is hold if and only if

$$X_t \leq \frac{-1}{7}$$

And since

$$\frac{(dEV(X_t))}{E(V(X_t))} = (28x_t^2 + 4x_t \ )dt$$

$$lnE(V(X_t)) = \int_0^t (28x_s^2 + 4x_s \ )ds$$

$E(V(X_t)) = \exp\left(\int_0^t (28x_s^2 + 4x_s \ )ds\right)$, then

$\lim_{t\to\infty} E^2\{X_t\} = \lim_{t\to\infty} \exp\left(2\left(\int_0^t (28x_t^2 + 4x_t \ )ds\right)\right) \neq 0$, then the stochastic differential equation is not asymptotically stochastically stable.

4. CONCLUSION AND FUTURE WORKS:

The study and find the stochastic stability is much richer than the classical stability of the ordinary differential equations. we know that the trivial solution is said to be stable if the derivative of Lyapunov function is less than or equal to zero, while if it is only negative-definite then it is asymptotically stable. That standard knowledge is also used in the stochastic differential equations. Just for the stability of stochastic differential equation we replace that's derivative$\dot{\boldsymbol{V}}^{\cdot}(\boldsymbol{X_t})$ by the function $\boldsymbol{LV}(\boldsymbol{X_t})$ which is obtained by applying stratonovich- formula to find the solution of the stochastic differential equation for the quadratic Lyapunov function. we explain the stability condition for linear and nonlinear stochastic differential equation by using the direct method (Lyapunov direct method), also we explain asymptotically stable in the large not almost this condition is satisfying, that is if the trivial solution is asymptotically stable but not asymptotically stable in large by the fact if the limit is not equal to zero. We explain the methods by several examples.

As a future studies one can study the stability (direct method) for some non-linear (harmonic or exponential) stochastic differential equation by using Stratonovich formula for their solution compare it with Ito formula

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